# Symmetries in Physics

## 1. Introduction

- **Symmetry**: is an operation that leaves an object invariant.
- Symmetry translates into a set of powerful mathematical constraints on the physical system.

## 2. Group Theory

### Definition: Group

A group \((G, \circ)\) is a set of elements (objects) \(\{a, b, c, \ldots\}\) endowed with a composition law \(\circ\) that has the following properties [called "group axioms"]:

- **G1: Closure**: For all \(a, b \in G\), the element \(c = a \circ b \in G\).
- **G2: Identity element** \(e\)
  \[\exists e \in G : e \circ a = a = a \circ e, \forall a \in G\]
- **G3: Inverse element** \(a^{-1}\) of \(a\)
  \[\forall a \in G, \exists a^{-1} \in G : a \circ a^{-1} = a^{-1} \circ a = e\]
- **G4: Associativity**: For all \(a, b, c \in G\), it holds \(a \circ (b \circ c) = (a \circ b) \circ c\).

### Examples

#### (a) Integer Numbers Group \(\mathbb{Z}_n\):

- **Addition** \((\mathbb{Z}_n, +)\) with \(n\) as modulus.
- **Inversion**: \(\text{mod } n = \text{modulo } n\), where integers (elements of \(\mathbb{Z}_n\)) are combined. If the result exceeds \((n-1)\), then subtract off \(n\) or multiple of \(n\) until the result is in \(\mathbb{Z}_n\).

#### (b) Permutation Group: \(S_3\)

\[S_3 = \{e, (123), (132)\}\]

- **Object** = \((abc)\)

Then:

- \(a_1 \equiv (123) = (12)(23)\) permute \(3 \rightarrow 2 \rightarrow 1\)
- \(a_2 \equiv (321) = (32)(12)\) permute \(1 \rightarrow 2 \rightarrow 3\)
- \(a_3 \equiv (123)(abc) = (123)(abc)\) permute \(1 \rightarrow 2 \rightarrow 3\)

Therefore:

- \(a_3a_1 = (abc)(123) = e \equiv (abc)\)

Each element \(i\) is its own inverse, i.e., \(a_i^2 = e\) for all \(a_i\).
**C**\(_n\) group:

- \((C_n, \text{successive applications}) = \text{group of } n \text{ element which are represented by rotations of a regular } n\text{-sided polygon, where rotations are through angles: } \frac{2\pi r}{n}, r = 1, 2, \ldots, n-1\).\n
\[C_n = \{e, c, c^2, \ldots, c^{n-1}\}\]

- \(c = e^{\frac{2\pi}{n}}\)
- \(c^\ell = e^{\frac{2\pi \ell}{n}}\)

**Order:**

The order of an element, \(a \in G\), is the power to which \(H\) is raised (no. of times \(H\) is applied) to obtain the identity element \(e\) of the group \(G\); i.e.,

\[a^s = e \Rightarrow s = \text{order of } a\]

**Note:** \(C_n\) is generated by one element, \(c\).

So we write \(C_n = \langle c \rangle\)  

**Definition:** Isomorphism:

Isomorphism means there is a one-to-one relationship between elements of two different groups.

Two groups \(G_1\) and \(G_2\) are isomorphic to each other if \(G_1 \cong G_2\) and their elements \(g_1 \leftrightarrow g_2\), \(i \in \{1, 2, \ldots, n-1\}\)

**Dihedral group** \(D_n\)

- \((D_n, \text{successive application}) = \text{symmetry group} \) of the regular polygon with \(n\) sides.

**e.g.:** \(D_3\)

\[D_3 = \{e, c, c^2, b, b_1, b_2\}\]

- \(c = \text{Rot}_{\frac{2\pi}{3}}\)
- \(b = \text{Rot}_{\frac{2\pi}{3}}\)
- \(b_1 = \text{Rot}_{\frac{2\pi}{3}}\)
- \(X_1 = X_2 = Y, X_3 = Z\)

**NB:** in terms of permutation

\[
\begin{align*}
(A B C) & \quad A \rightarrow B \rightarrow C \rightarrow A \quad (A B) (1)
\end{align*}
\]

- \(C \cong (123) \cong (231) \cong \text{sym} (123)\)

**Equivalence Relations**

- \(b_2 = cb_1 c^{-1}\)
- \(b_2 = b_1 c\)
- \(b_3 = c b_1 c\)
- \(b_3 = b_1 c^{-1} b_1 c\)
- \(b_3 = b_1 c^{-1} b_1 c^2\)

\(b_1 \cong b_2 \in D_3 = \{e, c, c^2, b, b_1, b_2\}\)
2.2. Properties of Groups.

2.2.1. Conjugation & Equivalence.

- **Definition:** Conjugation
  Two elements $a, b \in G$ are said to be conjugate if $a = g \cdot b \cdot g^{-1}, g \in G$.
  $g$ is called the "conjugating element." Need not be unique.

  \[ c = b \cdot c \cdot b^{-1} \quad \text{for } D_3 \]

- **Definition:** Equivalence Relation
  A relation $\sim$ between two elements $a, b \in G$ is said to be an equivalence relation if:
  \[(i) \text{ Reflexive: every element should be equivalent to itself} \]
  \[ a \sim a \quad \forall a \in G \]
  \[(ii) \text{ Symmetry: } a \sim b \Rightarrow b \sim a \]
  \[(iii) \text{ Transitive: } a \sim b \text{ and } b \sim c \Rightarrow a \sim c \]

  Conjugation is an equivalence relation since
  \[a \sim b \Leftrightarrow a = g \cdot b \cdot g^{-1} \]
  \[g \cdot e \cdot g^{-1} = g \Rightarrow (g \cdot a \cdot g^{-1}) \cdot (g \cdot a \cdot g^{-1})^{-1} = a \]
  \[a \sim a \text{ Reflexive} \]
  \[a \sim b \Leftrightarrow a = g \cdot b \cdot g^{-1} \]
  \[g^{-1} \cdot g \cdot e \cdot g^{-1} = b \] for $g^{-1} \cdot g \cdot e \cdot g^{-1} \in G$ by inverse.
  \[b \sim a \] with conjugate element $g^{-1}.

\[ (c) a \sim b \Leftrightarrow a = b \cdot c \cdot b^{-1} \]

- **Definition:** Subgroup
  A subgroup $H$ of a group $G$ is a set of elements $\{h\} \in G$, which is closed under the group composition law, i.e., a subgroup is some subset of group $G$ that form a group on its own.

\[ c = \frac{2 \pi}{n} \]

\[ c = 2 \cdot \text{fold rotation} \] for $n = 2, c^2 = e$. 

\[ c^2 = e \quad \text{for } D_3 \]
2.2. Conjugacy Classes:

Definition: Conjugacy Class
A conjugacy class of an element \( a \in G \), denoted as \( (a) \), is the set of elements \( \{ b \} \subseteq G \) which are conjugate to \( a \):

\[
(a) = \{ b \mid b = g a g^{-1}, b, g \in G \}
\]

Conjugacy classes are a special case of equivalence classes:

\[
(a) = \{ b \mid b \sim a, b \in G \}
\]

Any equivalence relation partitions a group into disjoint sets of elements, these sets are the equivalence classes. i.e., it's a decomposition of the group.

Example:

\[
C_n = \{ e, c, \ldots, c^{n-1} \} \quad \text{Abelian}
\]

\[
\forall g, c \in C_n : ag = g a
\]

\[
\forall b \in C_n \Rightarrow b = gag^{-1} = a g g^{-1} = a
\]

...every element is conjugate to itself only.

\[\Rightarrow (e) = \{ e \} \quad \Rightarrow G_n = (e) U (c) U \cdots U (c^{n-1}) \quad \exists \quad \text{conjugate classes.} \]

2.2.3. Subgroups:

Definition: Subgroup
A Subgroup \( H \) of a group \( G \) is a subset of \( G \) which forms a group under the composition law of \( G \).

For finite groups, it is sufficient to check closure of \( H \) and \( H \) to be a subgroup.

\[\exists \text{other 3 axioms follow from group } G\]

Proof:

\[\text{If } h_1 \in H \text{ chain } h_1^n, h_1^{-n} \in H\]

\[G_2: \text{ If } r = \text{order of } h_1 \Rightarrow h_1^r = e\]

\[G_3: h_1^r = h_1 h_1^{-1} = e \Rightarrow h_1^{-1} = h_1^{-r}\]

\[G_4: (h_1 h_2) \cdot h_3 = h_1 (h_2 h_3) \quad \text{sinn } \{ h_1 \}_{1, n} \subseteq G \]
Proper Subgroup:
A proper subgroup, \( H \), is a subgroup that is not the identity subgroup nor whole group \( G \).

\[ H \neq \{e\} \land H \neq G \]

2.2.4. Cosets:

Definition: Coset

Let \( H = \{h_1, h_2, \ldots, h_r\} \) be a proper subgroup of \( G \). For a given \( g \in G \), the sets

\[ gH = \{gh_1, gh_2, \ldots, gh_r\} \]

\[ Hg = \{h_1g, h_2g, \ldots, h_rg\} \]

are called **left** and **right cosets of** \( H \).

**Collary:**
There is an equivalence relation between an element, \( g \), and members of its coset:

If \( a \in gH \) and \( a \cdot b \)

\[ b \in gH \]

Proof:

(i) Reflexive: \( a \cdot a \Rightarrow \) every element is in its own coset.

If \( H = \{e, h_2, \ldots, h_r\}, \ a \in G \)

\[ \Rightarrow \ a \cdot a = \{a \cdot h_2, \ldots, a \cdot h_r\}, \ \text{has} = \{a, a \cdot h_2, \ldots, a \cdot h_r\} \]

\[ \Rightarrow a \in a \cdot a \text{ of } a \]

(ii) Symmetry: If \( a \cdot b \) \Rightarrow \( b \cdot a \)

\[ \{a \in \text{coset of } b \} \Rightarrow b \in \text{coset of } a \}

\[ a \cdot b \Rightarrow a = b \cdot h_k \in bH \]

\[ H = \{e, h_2, \ldots, h_r\}, \ bh_k = \{b, bh_2, \ldots, bh_r\} \]

\[ \Rightarrow b \cdot a \]

(iii) Transitive:
If \( a \cdot b \land b \cdot c \Rightarrow a \cdot c \)

\[ a \cdot b \Rightarrow a = b \cdot h_k \land b \cdot c \Rightarrow b \cdot c = c \cdot h_j \in cH \]

\[ \Rightarrow a = b \cdot h_k = (c \cdot h_j) \cdot h_k = c \cdot (h_k \cdot h_j) \in \text{closure} \]

\[ \Rightarrow a \cdot c \]

**Lagrange's theorem:**

If \( g_1H \) and \( g_2H \) are two (left) cosets of \( H \), then either \( g_1H = g_2H \) or \( g_1H \cap g_2H = \emptyset \)

A coset can be labelled by any of its elements since they are equivalent.

Proof:

Let \( gH = \{gh_1, gh_2, \ldots, gh_r\}, \ g \in G, H \subset G \)

Then coset of \( (gh_k) \) is:

\[ k \in \{1, r\} \]

\[ gh_kH = \{gh_kh_2, \ldots, gh_kh_r\} \]

but \( h_kH \subseteq H \)

In fact \( h_kH = H \)

\[ \Rightarrow gh_kH = gH \]

**Coset decomposition:**

If \( H \) is a proper subgroup of \( G \), then \( G \) can be decomposed into a union of (left) cosets of \( H \):

\[ G = H \cup g_1H \cup g_2H \cup \ldots \cup g_nH \]

where

\( g_1, g_2, \ldots, g_n \in G \) and \( g_1 \notin H, g_2 \notin H, \ldots, g_n \notin H \), etc.

The number \( n \) is called the *index of \( H \) in \( G \).*
2.2.5. Normal Subgroup

* Definition: Conjugate to $H$

If $H$ is a subgroup of $G$, then the set
\[ H^g = gHg^{-1} = \{ ghg^{-1}, gh_2g^{-1}, \ldots, gh_ng^{-1} \} \]
for a given $g \in G$, is called $g$-conjugate to $H$ or simply conjugate to $H$.

* Definition: Normal Subgroup

If $H$ is a subgroup of $G$ and
\[ H = gHg^{-1} \quad \forall g \in G \]
then $H$ is called a normal subgroup of $G$.

2.2.6. Quotient Group: $G/H$

* Definition: Quotient group

Let
\[ G/H = \{ Hg_1, Hg_2, \ldots, Hg_n \} \]
when $g_1, \ldots, g_n \in G$; $g_1H, g_2H, g_3H, \ldots$, etc.
be the set of all distinct cosets of $H$ in $G$.

If $H$ is a normal subgroup of $G$ with the multiplication law:
\[ (g_iH) \cdot (g_jH) = (g_i \cdot g_j)H \]
where $g_iH, g_jH \in G/H$. Then $(G/H, \cdot)$ is a group and is termed Quotient group.

Proof:

Go: Closure.
\[ \forall g_1, g_2 \in G, \quad g_1H \cdot g_2H \in G/H \]
\[ (g_1H) \cdot (g_2H) = (g_1 \cdot g_2)H = g_3H \in G/H \]

but $g_3 \in G$ by closure of $G$.

G1: Associative.
\[ \forall g_1, g_2, g_3 \in G, \quad (g_1H) \cdot (g_2H) \cdot (g_3H) = (g_1 \cdot (g_2 \cdot g_3))H \]
\[ = (g_1 \cdot g_2 \cdot g_3)H \]
\[ = (g_iH) \cdot (g_jH) \cdot (g_kH) \]
\[ (C_n, \cdot) \]
G2. Identity. For \( e \in G \), identity of \( G \), \( \forall g \in G \):
\[
(eH) \cdot (gH) = (e \cdot g)H = gH = (gH) \cdot (eH) = (eH) \cdot (gH)
\]
\[
\Rightarrow (eH) \text{ is identity of } G/H. \text{ [or simply } H \text{ as } eH = H\]

G3. Inverse. \\
\( \forall g, g^{-1} \in G : g \cdot g^{-1} = e \)
\[
\Rightarrow (gH) \cdot (g^{-1}H) = (g \cdot g^{-1})H = eH
\]
\[
\Rightarrow \text{ inverse of } (gH) \text{ is } (g^{-1}H).
\]

Uniqueness of multiplication law

\( \forall g, g_0, g_1 \in G \) \( \Rightarrow \) \( \forall h, h_0, h_1 \in H \) \( \text{ then} \)
\[
g_0 H = \{g_0 h_0, \ldots, g_0 h_1\} = g_0 h_0 H
\]
\[
g_1 H = \{g_1 h_0, \ldots, g_1 h_1\} = g_1 h_0 H
\]
\[
\Rightarrow (g_0 H) \cdot (g_1 H) = (g_0 g_1 H) \quad (2)
\]
\[
= (g_0 h_0 g_1 b_0) H \quad (3)
\]

\( \text{ does } (2) = (3) \)?

Closure of \( H \) \( \Leftrightarrow \) \( k \in H \) \( \Rightarrow \) \( kH = H \), \( \forall k \in G \)
(3) into (2):
\[
(g_0 h_0 g_1 b_0) H = (g_0 h_0 g_1 b_0) H = g_0 h_0 b_0 H
\]
\[
= g_0 h_0 b_0 H = g_0 h_0 b_0 H \quad \text{ always}
\]

\[\text{Definition: Direct product} \]

A group \( G \) can be expressed as a direct product (\( \times \)) of its subgroups
\( A \times B \), written \( G \cong A \times B \), iff

(\( \ast \)) All elements \( a \in A \) that \( B \) commute with

all elements of \( A \):

\( \forall b \in B, a \in A : ab = ba \)

\( \ast \) Every element \( g \in G \) can be expressed in a unique way as \( g = a_i b_j \), \( a_i \in A \) and \( b_j \in B \)

Consequences:

(a) \( A \) and \( B \) are normal subgroups of \( G \).

Proof:

\( A \) is normal subgp \( \Leftrightarrow \forall g \in G : gAg^{-1} = A \)

From (a): \( \forall g \in G : g = a_i b_j \) \( ; a_i \in A \) \( \land \) \( b_j \in B \)

\( \forall a \in A : \)
\[
gag^{-1} = a_i b_j a_i^{-1} \quad \text{From (i) } \Rightarrow \quad b_j a_i^{-1} a_i = a_i a_i^{-1} \quad \text{ closure} \]
\[
\Rightarrow a_i a_i^{-1} a_i a_i^{-1} a_i = A
\]
\[
\therefore A \text{ is a normal subgroup.}
\]

(b) \( \exists \) \( \forall \): \( \therefore \)

\( G \cong A \times B \Rightarrow G/A \cong B \)

Proof:

\( G/A = \{g_i A \} \quad g_i \in G \land g_i = e \)

(\( \ast \)) \( \forall \)
\[
g = a_k b_i \quad \Rightarrow \quad b_i a_k \quad a_k \in A \land b_i \in B
\]

So \( \forall g, A \in G/A : gA = b_i a_k A \Rightarrow b_i A \)

\( \therefore \) every coset in \( G/A \) is identified with an element in \( b_i \in B : q_i A \leftrightarrow b_i \)

Multiplication law is obeyed by cosets \( b_i A \):

\( \forall b_i, b_j \in B : (b_j A), (b_i A) = (b_i b_j) A \in B \)
there is one-to-one correspondence between elements of \( G/A \), \( B \):

\[
g_0 \circ A \leftrightarrow b_j \cdot A
\]

\[
\frac{(g_0 \circ g_m) \cdot A}{g_0} \leftrightarrow \frac{(b_j \circ b_k) \cdot A}{b_k}
\]

\[
G/A \cong B \quad \iff \quad G \cong A \times B
\]

* Example:

\[
C_6 = \{e, c, c^2, c^3, c^4, c^5\}, \quad c = \frac{2\pi}{6} = \frac{\pi}{3}
\]

Normal subgroups:

\[
C_2 = \{e, c^3\}, \quad c^3 = 1 \iff c^2 = 1
\]

\[
C_3 = \{e, c^2, c^4\}
\]

- Recall:

\[
\forall g \in C_2: \quad g \circ g^{-1} = e, \quad g \circ g = g
\]

- Recall:

\[
\forall g \in C_3: \quad g \circ g^{-1} = e, \quad g \circ g = g
\]

- Does \( C_4 \cong C_2 \times C_2 \)?

\[
C_4 = \{e, c, c^2, c^3\}
\]

\[
\text{normal subgroups: } C_2 \times C_2 = \{e, c, c^2, c^3\}
\]

\[
\forall g_1, g_2 \in C_2, \forall h_1, h_2 \in C_2:
\]

\[
since C_2 \times C_2 \text{ is abelian}
\]

\[
\forall g \in C_4:
\]

\[
f = g \circ h, \quad g, h \in C_2, \ b, h \in C_3
\]

\[
c = c, \quad c^2 \cdot c = c^3, \quad c^4 = c^2
\]

\[
does \ C_4/\text{C}_3 \cong C_2?
\]

\[
C_4/C_3 = \{g_3 \cdot C_3\}, \quad g_3 \in C_4
\]

\[
e \cdot C_3 = \{e, c, c^2, c^3\} \cong E
\]

\[
c \cdot C_3 = \{c, c^2, c^3\} \cong B
\]

\[
c^2 \cdot C_3 = \{c^2, c^4, e\} \cong B
\]

\[
\therefore C_4/C_3 = \{E, B\}
\]

\[
BE = c \cdot C_3 \circ C_3 = (c \cdot c) \cdot C_3 = c \cdot C_3 = B
\]

2.3. Morphisms between Groups

Definition: Group Homomorphism

A Homomorphism is a mapping, \( f \), from one set, \( A \), to another set, \( B \), written

\[
f: A \rightarrow B
\]

preserving some structure or other.

Definition: Group Homomorphism

If \( (A, \cdot) \) and \( (B, *) \) are two groups, then group Homomorphism is a function \( f \) from the set \( A \) into the set \( B \), i.e., each element of \( A \), \( a \in A \), is mapped into a single element \( b = f(a) \in B \), such that the following multiplication law is preserved:

\[
f(a, b) = f(a) * f(b).
\]

\[
f: A \rightarrow B
\]

\[
a \rightarrow b' = f(a)
\]

\[
a \cdot a' \rightarrow b = f(a) \circ b
\]

Elements of \( B \) can either be the image of:

- A single element in \( A \)
- Many elements of \( A \)
- No element of \( A \)
Elements of $A$ can have either:
- no image in $B$
- only one image in $B$

In general:
\[ f(A) \neq B \iff f(A) \subseteq B \]

$f(A)$ is a subgroup of $B$

\[ \forall a_1, a_2 \in A, \quad f(a_1 \cdot a_2) = f(a_1) \cdot f(a_2) \]

by closure

\[ \forall a_1, a_2 = a_3 \in A \]

\[ f(a_1) = f(a_3) \]

$\in B \implies \in B$

### Definition: Group Isomorphism

Let a 1:1 mapping $f$ of $(A, \cdot)$ onto $(B, \ast)$, such that each element of $a \in A$ is mapped into a single element of $b = f(a) \in B$, and conversely, each element of $b \in B$ is the image of a single element of $a \in A$.

If this bijective 1:1 mapping satisfies the composition law

\[ f(a \cdot a) = f(a) \ast f(a) \]

It is said to define an isomorphism between the groups $A$ and $B$, $\alpha$ is denoted by $A \cong B$.

### Endomorphism

A group homomorphism of $A$ into itself.

### Automorphism

A group isomorphism of $A$ into itself.

### 2.3.1 The Kernel of group Homomorphism

#### Definition: Kernel

A kernel of a group Homomorphism, or Homomorphism, or mapping $f$, denoted $\ker f$, is the set of elements in $A$ which are mapped into the identity element of $B$:

\[ \ker f = \{ a \in A \mid f(a) = e_B \land e_B \in B \} \]

$\ker f$ forms a Normal subgroup $\ker f$, of group $A$.

**Proof:**

Go: closure. \( \forall a_1, a_2 \in \ker f \):

\[ f(a_1 \cdot a_2) = f(a_1) \ast f(a_2) \]

\[ = e_B \ast e_B \]

\[ = e_B \implies (a_1, a_2) \in \ker f \]

G1: Associative. \( \forall a_1, a_2, a_3 \in \ker f \):

\[ f(a_2 \cdot (a_3 \cdot a_2)) = f(a_2) \ast f(a_3 \cdot a_2) \ast f(a_2) \ast f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \cdot f(a_3) \cdot f(a_2) \end{equation}

**G2:** Identity. \( \forall a_1 \in \ker f \):

\[ f(a_1 \cdot e_A) = f(a_1) + f(e_A) \]

\[ = f(a_1) + e_B \]

\[ \implies f(a_1) = e_B \implies e_A \in \ker f \]

**G3:** Inverse. \( \forall a_2 \in \ker f, \exists a^{-1}_2 \in A \):

\[ f(a_2 \cdot a^{-1}_2) = f(a_2) \ast f(a^{-1}_2) \]

\[ = f(a_2) \ast e_B \]

\[ \implies f(a^{-1}_2) = e_B \implies e_B \in \ker f \]

Now need to show that:

\( \forall a \in A \):

\[ a \in \ker f \ast = \ker f \]
3. Group Representations (Reps)

Definition: Vector Space \( V \)

A vector space \( V \) over the field of complex numbers \( \mathbb{C} \) is a set of elements \( \{ \mathbf{v} \} \), endowed with composition laws (operations), \((+,-)\) satisfying the following properties:

(a) Closure: \( \mathbf{v} + \mathbf{w} \in V \) \( \forall \mathbf{v}, \mathbf{w} \in V \)

(b) Commutativity: \( \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \) \( \forall \mathbf{v}, \mathbf{w} \in V \)

(c) Associativity: \( (\mathbf{v} + \mathbf{w}) + \mathbf{z} = \mathbf{v} + (\mathbf{w} + \mathbf{z}) \) \( \forall \mathbf{v}, \mathbf{w}, \mathbf{z} \in V \)

(d) Identity (null) vector.

\[ \exists \mathbf{0} \in V : \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v} \] \( \forall \mathbf{v} \in V \)

(e) Existence of Inverse.

\[ \forall \mathbf{v} \in V, \exists (-\mathbf{v}) \in V : \mathbf{v} + (-\mathbf{v}) = \mathbf{0} \]

(b0) \( \lambda \mathbf{v} \in V \) \( \forall \lambda \in \mathbb{C} ; \forall \mathbf{v} \in V \)

(b1) \( \lambda (\mathbf{v} + \mathbf{w}) = \lambda \mathbf{v} + \lambda \mathbf{w} \) \( \forall \lambda \in \mathbb{C} ; \forall \mathbf{v}, \mathbf{w} \in V \)

(b2) \( (\lambda + \lambda_2) \mathbf{v} = \lambda \mathbf{v} + \lambda_2 \mathbf{v} \) \( \forall \lambda, \lambda_2 \in \mathbb{C} ; \forall \mathbf{v} \in V \)

(b3) \( \lambda \cdot (\mathbf{v} + \mathbf{w}) = (\lambda \cdot \mathbf{v}) + (\lambda \cdot \mathbf{w}) \)

(b4) \( \mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v} \) \( \forall \mathbf{v}, \mathbf{w} \in V \)

Example: Group elements as matrices

Rotation: \( \phi = \phi + \beta \)

\[ \begin{align*}
\mathbf{x} &\rightarrow \mathbf{x}' = r \sin \beta \cos (\phi + \beta) \\
\mathbf{y} &\rightarrow \mathbf{y}' = r \cos \beta
\end{align*} \]

\[ \begin{align*}
\mathbf{z} &\rightarrow \mathbf{z}' = z
\end{align*} \]

For \( \mathbf{C}_3 = \{ e, \alpha, \alpha^2 \} \) \( \subset \mathbb{S}(2, \mathbb{R}) \)

\[ e \mapsto D(e) = D(2\pi) = \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix} \]

\[ \alpha \mapsto D(\alpha) = D(\frac{2\pi}{3}) = \begin{pmatrix} 0 & -1 \\
1 & 0 \end{pmatrix} \]

\[ \alpha^2 \mapsto D(\alpha^2) = D(\frac{4\pi}{3}) = \begin{pmatrix} -1 & -\sqrt{3} \\
\sqrt{3} & -1 \end{pmatrix} \]

Note:

\[ D(e) D(\alpha) = D(\alpha) D(e) \]

\. Definition: Group Rep

A group representation, \( D \), of dimension \( N \), \( D : G \rightarrow GL(N, \mathbb{C}) \)

\[ y \rightarrow D(y) \]

is a homomorphism of the elements \( y \) of a group \( (G, \cdot) \) into the group \( GL(N, \mathbb{C}) \) of non-singular linear
transformations of a vector space \( V \) of dimension \( N \), i.e., the set of \( N \times N \) dimensional invertible matrices in \( \mathbb{C}^N \).

Homomorphism [a Group Homomorphism] implies that the group multiplication is preserved:

\[ D(a, g_1) \circ D(a, g_2) = D(a, g_1 g_2) \quad \forall a, g_1, g_2 \in G \]

For \( \forall M \in \text{GL}(N, \mathbb{C}) : \det(M) \neq 0 \).

Rotation matrices are orthogonal as they preserve the length of the rotated vector.

\[ x \mapsto x' = Rx \]

\[ x^T x' = x^T R^T Rx = x^T x \]

provided \( R^T R = I \Rightarrow R^T = R^{-1} \)

3.1. Induced transformation of the wavefunction in QM:

For an 2p state in H-atom:

\[ |n\ell m\rangle \propto \cos \beta \]

\[ \langle n\ell m| = \langle n\ell m| \]

\[ \{ n\ell (x^1) = n(x) \]

\[ x' = R x \]

\[ \Rightarrow \begin{bmatrix} n(x') \end{bmatrix} = n(R^{-1} x) \]  \( \text{(4)} \)

note R is independent of x.

For H-atom:

Hamiltonian:

\[ \hat{H} = \frac{\hat{p}^2}{2m} + V(r) \]

eigenfunctions (state wave):

\[ U_{n\ell m}(\theta, \phi) = |n\ell m\rangle \]

Rotating \( U_{n\ell m} \) about \( z \)-axis \( \Rightarrow U_{n\ell m}^z, \) which

\[ \text{is a superposition of odd } U_{n\ell m}, \text{i.e.,} \]

\[ U_{n\ell m}^z (\theta, \phi) = \sum_{m' = -\ell}^{\ell} C_{m'\ell m} U_{n\ell m'} (\theta, \phi) \]  \( \text{(5)} \)

\[ m' \in \mathbb{Z}, \theta, \phi \in \mathbb{R} \Rightarrow D_{m'\ell m} = (2\ell + 1)x(2\ell + 1) \]

rep. of this rotation.

Proof:

\[ \text{Rep } \Rightarrow \text{ Homomorphism} \]

\[ D(a, g_1) = D(a) D(g_1) \]

\[ U_{n\ell m} = \sum_{m'} D_{m'\ell m} U_{n\ell m'} \]

\[ = \sum_{m'} D_{m'\ell m} (\theta) \sum_{m''} D_{m''m'} (\phi) U_{n\ell m''} \]

\[ = \sum_{m''} D_{m''m'} (\theta) D_{m''m} (\phi) U_{n\ell m''} \]

\[ \Rightarrow \]

\[ U_{n\ell m} = \sum_{m'} D_{m'\ell m} (\theta) U_{n\ell m'} \]

\[ = \sum_{m'} D_{m'\ell m} (\theta_1) \sum_{m''} D_{m''m'} (\phi_2) U_{n\ell m''} \]

Example:

\[ n = 2, \ell = 1, m = -1, 0, 1. \]

\[ U_{210} = F(r) \ Y_{210}(\theta, \phi) \]

\[ Y_{210} = -\sqrt{\frac{3}{2\pi}} \ \cos \theta \ \sin \phi \ e^{i\phi} \]  \( \text{(6)} \)

\[ Y_{10} = \sqrt{\frac{3}{4\pi}} \ \cos \theta \ \sin \phi \]

\[ Y_{11} = \sqrt{\frac{3}{4\pi}} \ \sin \theta \ e^{i\phi} \]

From (4):

\[ U_{210} = U_{210} \ (r^2 \ \sin \theta) \]

\[ R \ \sin \theta \ x = \left( \begin{array}{c} \cos \beta \ -\sin \beta \\ \sin \beta \ \cos \beta \end{array} \right) \ (x) = \left( \begin{array}{c} x' \\ z' \end{array} \right) \]

\[ z' = \sin \beta \ x + \cos \beta \ z \]

\[ z' = \sin \beta \ [\sin \theta \ e^{i\phi} + \cos \theta \ e^{-i\phi}] \quad \text{if} \quad \beta = 2 \]

Using (2):

\[ \sin \beta \frac{A}{2} \ (U_{211} - U_{211}) + \sin \beta \ U_{20} \]  \( \text{(3)} \)
but \( z' = \cos \theta' = U_{140} \)

\[ U_{210} = \sin \theta \frac{1}{\sqrt{2}} (U_{21-1} - U_{211}) + \cos \theta U_{210} \]

Using (a):

\[ D_{00} = \cos \theta, \quad D_{01} = -D_{0-1} = \frac{1}{\sqrt{2}} \sin \theta \]

\[ x' = \cos \theta x - \sin \theta z = \cos \theta \left[ \sin \theta \cos \phi - \sin \theta \sin \phi \right] \]

\[ = \cos \theta \left( \frac{1}{\sqrt{2}} (U_{21-1} - U_{211}) \right) - \sin \theta U_{210} \]

\[ x' = \sin \theta \cos \phi = \frac{1}{\sqrt{2}} \left( U_{21-1} - U_{211} \right) \]

\[ \Rightarrow \frac{1}{\sqrt{2}} \left[ U_{21-1} - U_{211} \right] = \frac{1}{\sqrt{2}} (\cos \theta (U_{21-1} - U_{211}) - \sin \theta U_{210} \quad (a) \]

\[ y' = y \]

\[ \Rightarrow U_{211} + U_{211} = U_{21-1} + U_{211} \quad (b) \]

(a) and (b):

\[ U_{211} = \frac{1}{2} (\cos \theta + 1) U_{211} \]

\[ + \frac{1}{2} (\cos \theta - 1) \sin \theta U_{210} \]

\[ + \frac{1}{2} \left( 1 + \cos \theta \right) U_{211} + \frac{1}{2} \sin \theta U_{210} \]

\[ \Rightarrow D_{21} = \frac{1}{2} (1 + \cos \theta) \quad D_{2-1} = \frac{1}{2} (1 - \cos \theta) \]

\[ D_{10} = -\frac{1}{\sqrt{2}} \sin \theta \quad D_{0-1} = \frac{1}{\sqrt{2}} \sin \theta \]

\[ D_{11} = \frac{1}{2} (1 - \cos \theta) \quad D_{1-1} = \frac{1}{2} (1 + \cos \theta) \]

\[ D_{m0} = \begin{pmatrix}
\cos \theta & \frac{1}{2} \sin \theta & \frac{1}{2} \sin \theta \\
\frac{1}{2} \sin \theta & \frac{1}{2} (1 + \cos \theta) & \frac{1}{2} (1 - \cos \theta) \\
\frac{1}{2} \sin \theta & \frac{1}{2} (1 - \cos \theta) & \frac{1}{2} (1 + \cos \theta)
\end{pmatrix} \]

3.2. Equivalent Reps

Definition: Equivalent Reps

Two Reps \( D_1 \) and \( D_2 \) are equivalent if there exists an isomorphism \( (1:1 \) correspondence) between \( D_1 \) and \( D_2 \). Such an equivalence is denoted as

\[ D_1 \cong D_2 \text{ or } D_1 \sim D_2. \]

To equivalent Reps may be related by a similarity transformation \( S \):

\[ D_1(g) = S D_2(g) S^{-1} \quad \forall g \in G \]

and \( S \) is independent of \( g \).

Proof:

If \( D_1 \) is Rep of group \( G \):

\[ D_1(g, g_2) = D_1(g) D_2(g_2) \quad \forall g_1, g_2 \in G \]

Let

\[ D_2(g) = S D_1(g) S^{-1} \quad \forall g \in G \]

\[ \Rightarrow D_2(g_1, g_2) = S D_1(g_1, g_2) S^{-1} \]

\[ = S D_1(g_1) D_2(g_2) S^{-1} \]

\[ = S D_1(g_1) S S^{-1} D_2(g_2) S^{-1} \]

\[ = S D_1(g_1) S^{-1} D(g_2) S^{-1} \]

\[ = D_2(g_1) D_2(g_2) \]

Main point is as long as we rotate by same angle it does not matter where the rotation is performed as far as the two space positions can be related.

\[ \text{If } \vec{r}' = D(g) \vec{r} \]

\[ S \vec{r}' = S D(g) \vec{r} = S D(g) S^{-1} S \vec{r} \]
Definition: Character

Character \( \gamma \) of a rep \( D \) of a group \( G \) is defined as the set of all traces of the matrices \( D(g) \):

\[
\gamma = \left\{ \gamma(g) : \gamma(g) = \sum \left[ D(g) \right]_{ii} \right\}, \quad g \in G
\]

Corollary:

Equivalent reps. have the same character.

Conversely, if two reps have the same character, they are equivalent.

Proof:

\[
D_1 \cong D_2 \iff D_1 = S D_2 S^{-1}
\]

\[
\gamma(\theta) = \text{Tr}(D_2) = \sum_i [D_2]_{ii}
\]

\[
\gamma(\theta) = \text{Tr} (S^{-1} D_2 S) = \text{Tr} (S D_2 S^{-1}) = \text{Tr} (D_1)
\]

3.3. Reducibility of Reps

Definition: Reducible Reps

A rep. of dimension \([m+n]\) is said to be reducible if \( D(g) \) can be written as

\[
D(g) = \begin{pmatrix}
A(g) & C(g) \\
0 & B(g)
\end{pmatrix}
\]

where

- \( A(g) \) is \([m \times m]\) matrix
- \( B(g) \) is \([n \times n]\) matrix
- \( C(g) \) is \([m \times n]\) matrix
- 0 is \([n \times m]\) null matrix

Then \( A(g) \) and \( B(g) \) constitute \([m]\) and \([n]\) -dimensional reps of Group \( G \).

Proof:

\[
D(\theta \theta') = \begin{pmatrix}
A(\theta \theta') & C(\theta \theta') \\
0 & B(\theta \theta')
\end{pmatrix}, \quad \forall \theta \theta' \in G
\]

\[
D(g) D(\theta) = \begin{pmatrix}
A(g) A(\theta) & A(g) C(\theta) + C(\theta) B(\theta) \\
0 & B(g) B(\theta)
\end{pmatrix}
\]

\[
\Rightarrow A(g \theta) = A(g) A(\theta)
\]

\[
B(g \theta) = B(g) B(\theta)
\]

Example:

3Din rep. of rotations about 2-axis:

\[
D^{(3)} = R = \begin{pmatrix}
\cos \beta & -\sin \beta & 0 \\
\sin \beta & \cos \beta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\cong \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}
\]

\[
D^{(3)} = A \text{ is 2-dim rep of rotations about 2-axis}
\]

\[
D^{(3)} = I \text{ is 1-dim rep }
\]

\[
\Rightarrow D^{(3)} = D^{(2)} \oplus D^{(1)}
\]

Definition: Completely Reducible Reps

A group rep. \( D(g) \) is said to be completely reducible, if there exists a non-singular matrix \( M \in \text{GL}(N, \mathbb{C}) \) independent of the group elements, such that

\[
M D(g) M^{-1} = \begin{pmatrix}
D_1(g) & 0 & \cdots & 0 \\
0 & D_2(g) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D_r(g)
\end{pmatrix}, \quad \forall g \in G
\]

\( D_1(g), D_2(g), \ldots, D_r(g) \) divide \( D \) into reps of lower dimensions, i.e.,

\[
\dim(D) = \sum_{i=1}^r \dim(D_i)
\]

and is denoted by the direct sum

\[
D = D_1 \oplus D_2 \oplus \cdots \oplus D_r
\]
\[ D(\sigma) = D_1(\sigma) \oplus D_2(\sigma) \oplus \cdots \oplus D_r(\sigma) = \sum_{i=1}^r D_i(\sigma) \]

**Definition:** Irreducible Reps. (Irreps)

A group rep. \( D(\sigma) \) which cannot be written as a direct sum of other reps is called irreducible.

Irreps are the basic blocks (matrices) of \( \text{GL}(N, \mathbb{C}) \). Any reducible rep can be decomposed as a direct sum of irreps.

### 3.4 Groups acting on vector spaces

- Group element, \( g \in G \), acts on set of objects, \( \{o_i\} \). Under Homomorphism:
  - \( g \rightarrow D(\sigma) : D(\sigma) = D(\sigma_1) D(\sigma_2) \)
  - \( o_i \rightarrow y D(\sigma) o_i \)
  - \( = g o_i \rightarrow D(\sigma) y o_i \)

### 3.4.1 Vector Space Axioms

- These are given under Chapter 8. [2 pages back] so no need to rewrite them again.

- Vector space under addition, \( (V, +) \), forms an abelian group.

- Example: Hilbert space [function space]. Consider the space of functions of real variable

\[ x \text{ over interval } x \in [0, 1] \text{ such that } f(0) = f(1) = 0. \]

\( f(\sigma) \) can be decomposed in terms of basis functions \( u_n \),

\[ f(x) = \sum_{n \in \mathbb{N}} f_n u_n(x). \]

\( f_n \) : Component of \( f \) in \( u_n \)-basis

Since the sum is infinite \( \Rightarrow \) space has infinite dimensions \( \Rightarrow \) called Hilbert space.

#### 3.4.2 Dimension of a Vector Space

- **Definition:** Linear Independent

A set of vectors \( \{e_i\}_{i=1}^m \) is said to be linearly independent if there is no non-trivial combination of them which yields the null vector:

\[ \sum_i \lambda_i e_i = 0 \Rightarrow \lambda_i = 0 \]

Only solution is 0.

- **Basis:** A linearly independent set of vectors \( \{e_i\}_{i=1}^N \) form a basis of the vector space iff they span the whole space:

\[ \forall x \in V : x = \sum_i v_i e_i \]

- **Dimension of a Vector Space** is the number of basis vectors.
3.4.3. Groups as Linear Transformations on a Vector Space

- Linear transformation:

\[ \tilde{T}(xu + \beta v) = x\tilde{T}u + \beta\tilde{T}v \]

\( \forall x, \beta \in \mathbb{C}; \forall u, v \in V; \forall \tilde{T} \in \text{GL}(V) \)

For \( \tilde{T} \) to be a rep. of a group then

\[ \tilde{T}(g) = \tilde{T}(g) \tilde{T}(e) \]

- Connection between \( LT, \tilde{T} \), and matrix space, \( \text{GL}(V, \mathbb{C}) \):

\[ \tilde{T}(g) \begin{pmatrix} D_1(g) \\ D_2(g) \end{pmatrix} = \text{equivalent matrix} \]

(i) Choose a basis \( \{ e_i \} \), such that

\[ v = u_i e_i, \forall u_i \in V \]

(ii) Act by \( \tilde{T} \) on \( u \) (vector):

\[ \tilde{T}u = (\tilde{T}e_i)u_i \]

\[ \tilde{T}e_i \in V \Rightarrow (\tilde{T}e_i) = \tilde{T}e_i = a_{ij} e_j \]

\[ \tilde{T}u = e_j a_{ij} u_i = \frac{u_j}{u_i} e_j = u_i' e_j \]

\( a_{ij} \) is the matrix which represents the action of \( LT, \tilde{T} \). It multiplies the coordinates in the same basis.

\[ e = \{ x, y, z \}; \tilde{T} = \begin{pmatrix} x_1' & y_1' & z_1' \\ \frac{x_2}{\sqrt{2}} & \frac{y_2}{\sqrt{2}} & \frac{z_2}{\sqrt{2}} \end{pmatrix} \]

\[ f = \begin{pmatrix} x \choose y \choose z \end{pmatrix} \]

\[ \tilde{T}f = \begin{pmatrix} x' \choose y' \choose z' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \choose y \choose z \end{pmatrix} \]

- Basis Reps. Relations:

- Relation between 2 matrix reps. of \( \tilde{T} \) in 2 diff. bases.

\[ \tilde{T} \text{ acts on vector basis } \{ e_i \}_{i=1}^N, \text{ which related to the basis } \{ e'_i \}_{i=1}^N, \text{ a action of } \tilde{T} \text{ on } \{ e_i \} \text{ is known:} \]

\[ \tilde{T}(f_i) = \tilde{T}(e_i')e_i = f_i' \]

where:

\[ e = e S \Rightarrow e' = e S^{-1} \]

\[ e_k' = f_k' S_{kk} \]

\[ \tilde{T}(e_i) = e_k D_{ki} \quad D = D(f) \quad v \text{ in } e \text{-basis} \]

\[ \tilde{T}(f_i) = f_k D_{ki} S_{ii} = f_k (S^{-1} D S)_{ki} \]

From (5), \( \tilde{T}(f_i) = f_k D_{ki} \quad \tilde{T}(e_i) = e_k D_{ki} S_{ii} \)

\[ \tilde{T}(e_i) = e_k D_{ki} S_{ii} \]

\[ D = S^{-1} D \tilde{T}(e) S \]

- Relation between components of a vector in e-basis and f-basis:

\[ \tilde{T}u = \sum_j u_j e_j = \sum_j u_j e_j' \]

\[ u = u f = u \tilde{T} \]

but:

\[ f = e S \]

\[ e S u = e u f = e u \tilde{T} \]

\[ u = S u \tilde{T} \Rightarrow u \tilde{T} = S^{-1} u \]

- Example:

\[ e = \{ x, y, z \}; f = \{ x_1, y_1, z_1 \} \]

\[ \tilde{T}f = \begin{pmatrix} x_1' \choose y_1' \choose z_1' \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \choose y \choose z \end{pmatrix} \]

[3]: dim Rotatn in e-basis.
3.5. Direct Product of Reps. and the Clebsch-Gordan Series

- Idea: Formalism that allows the addition of QM numbers [spin, any more, color, ...] of composite systems [2 or more quarks, leptons, bosons, ...]

- Let a composite system be made of 2 particles with wave functions \( \psi_a(x), \psi_b(x) \). Under rotation \( \hat{\gamma}(\theta) \), they transform on:

\[
\begin{align*}
\psi_a' &= D^{(a)} \psi_a = \gamma_a \psi_a \\
\psi_b' &= D^{(b)} \psi_b = \gamma_b \psi_b
\end{align*}
\]

where \( D^{(a)} \) and \( D^{(b)} \) are two different reps of group \( G \).

- The composite system transforms then as:

\[
\psi_c' = D^{(c)} \psi_c
\]

- Define \( \gamma_{ac} \equiv \gamma_a \gamma_c \)

\[
\gamma_{ac} = D^{(a)} D^{(c)} \gamma_{bd}
\]

- Let the ordered pairs:

\[
\begin{align*}
(ac) &\equiv A \\
(bd) &\equiv B
\end{align*}
\]

\[
\Rightarrow \begin{bmatrix}
\gamma_{ac}
\end{bmatrix} = D^{(ac)} \begin{bmatrix}
\gamma_{bd}
\end{bmatrix}
\]

with:

\[
\begin{align*}
D^{(ac)} &= D^{(a)} D^{(c)} \\
\text{direct product}
\end{align*}
\]
If $D^{(\alpha \times \beta)_G}$ is a rep. of Group $G$ as it satisfies the multiplication law:

$$D^{(\alpha \times \beta)_G}(g_1 g_2) = D^{(\alpha)_G}(g_1) D^{(\beta)_G}(g_2)$$

g_1 = g_1', g_2 = g_2'$

If there is no interaction of the particle, or

$D^{(\alpha \times \beta)_G}$ is an irrep. of direct product group $G \times G$.

$\left[ \text{indep. reltn. of } \alpha \& \beta \right]$

Symmetry of each particle is preserved.

If there is interaction, then $G \times G \to G$ and $D^{(\alpha \times \beta)_G}$ are reducible reps. can be written as direct sum of irreps. of $G$:

$$D^{(\alpha \times \beta)_G} = \sum_{\alpha} \alpha \otimes D^{(\gamma)}$$

called Clebsch-Gordan Series (decomposition).

Formally:

Definition: Direct Product of irreps
If $D^{(\alpha)}$ and $D^{(\beta)}$ are two irreps. of Group $G$, a direct-product, denoted as

$$D^{(\alpha \times \beta)}(g_1 g_2) = D^{(\alpha)}(g_1) \otimes D^{(\beta)}(g_2)$$

can be constructed as follows:

$$D^{(\alpha \times \beta)}_{ij, kl} = D^{(\alpha)}_{ik} D^{(\beta)}_{lj}$$

Frequently, direct products of irreps. are called Tensor Product.

It can be shown that $D^{(\alpha \times \beta)_G}$ is an irrep. of the (direct) product group $G \times G$.

1. Clebsch-Gordan Series:

If $g_1 = g_2 = g$, then the symmetry of the product group $G \times G$ is reduced to the diagonal $G$, i.e., $G \times G \to G$.

In this case, $D^{(\alpha \times \beta)}(g g)$ may not be an irrep. and can be further decomposed into a direct sum of irreps. of $G$:

$$D^{(\alpha \times \beta)}(g g) = \alpha \otimes D^{(\gamma)}(g) = \sum_{\alpha} \alpha \otimes D^{(\gamma)}(g)$$

Such a decomposition is called a Clebsch-Gordan Series, and the coefficients $[\alpha \gamma]$ are called Clebsch-Gordan Coefficients.

4. Continuous Groups

## Lie Groups

<table>
<thead>
<tr>
<th>Group</th>
<th>Proper H.</th>
<th>No. Indep. Params</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
<tr>
<td>$GL(N, \mathbb{C})$</td>
<td>$\det M \neq 0$</td>
<td>$2N^2$</td>
<td>General rep.</td>
</tr>
<tr>
<td>$SL(N, \mathbb{C})$</td>
<td>$\det M = 1$</td>
<td>$2(N^2 - 1)$</td>
<td>$SL(N, \mathbb{C}) \subset GL(N, \mathbb{C})$</td>
</tr>
<tr>
<td>$O(N, \mathbb{R})$</td>
<td>$\sum_{i=1}^{N} (x_i)^2 = \frac{N}{2}$</td>
<td>$\frac{1}{2} N(N+1)$</td>
<td>$O^T = O^{-1}$</td>
</tr>
<tr>
<td>$SO(N, \mathbb{R})$</td>
<td>above +</td>
<td>$\det O = 1$</td>
<td>$\frac{1}{2} N(N-1)$</td>
</tr>
<tr>
<td>$SU(N, \mathbb{C})$</td>
<td>$\sum_{i=1}^{N}</td>
<td>x_i</td>
<td>^2 = \sum_{i=1}^{N}</td>
</tr>
<tr>
<td>$SU(N, \mathbb{C})$</td>
<td>$\det U = 1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>


SO(N, M) \sum_{j=1}^{M} x_j^a \tilde{x}_j^a \quad \text{old} \Lambda^T g \Lambda = g

\delta_{ij} = \text{diag} \left( 1, 1, \ldots, 1, -1, \ldots, -1 \right)

\text{Proof:}

Assuming (i) and (ii):

\text{Tr} (\ln M) = \text{Tr} \left( s \ln \tilde{M} s^{-1} \right)

= \text{Tr} \left( s^{-1} s \ln \tilde{M} \right)

= \text{Tr} (\ln \tilde{M}) = \sum \ln M_{ij}

A. Useful Matrix Relations in GL(N, C)

\text{Definitions:}

(i) \quad e_M = \sum_{n=0}^N \frac{M^n}{n!}

(ii) \quad \ln M = \sum_{n=1}^N (-1)^{n+1} \frac{(M-1)^n}{n} = \int \text{di} (M-1) \left[ i(M-1)^{-1} \right]^N

where \quad M \in \text{GL} (N, C), s.t., \ det M \neq 0

\text{Basic properties:}

If \quad [M_1, M_2] = 0 \text{ and } M_{12} \in \text{GL}(N, C)

then

(i) \quad e_{M_1} \cdot e_{M_2} = e_{M_1 + M_2}

(ii) \quad \ln (M_1 M_2) = \ln M_1 + \ln M_2

\text{Identity:}

\ln (\det M) = \text{Tr} (\ln M)

This identity can be proven more easily if M can be diagonalized through a similarity

\hat{M} = S^{-1} M S

\text{and noting that:} \quad \ln M = S \ln \hat{M} S^{-1}

4.1. SO(N) Group

SO(N, R) is the Special \quad \det O = 1

[0: \text{rep. Orthogonal [O$^T = O^{-1}$] group of rotations through any angle in N dimensions over the field of Real numbers. It preserves the length of a vector.}]

$$\sum_{k=1}^{N} (x_k)^2 = \sum_{k=1}^{N} (x_k^2)^2.$$
4.1.1. $\text{SO}(2)$ group

- Transf. of a point, $P(x,y)$, under a rotation $\phi$ about $z$-axis:
  $$(x',y') = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} (x,y)$$
  $\equiv O(\phi)$.

$O(\phi)$ is the defining rep. of $\text{SO}(2)$ over $\mathbb{R}$.

- Note:
  $\det(O(\phi)) = 1$
  $O(\phi)^T O(\phi) = I_2$
  $O(\phi) O(\phi) = O(\phi + \phi) = O(\phi) O(\phi)$
  $\to$ $\text{SO}(2)$ is Abelian.

$O(\phi)$ is an irrep of $\text{SO}(2)$ over $\mathbb{R}$.

- Transf. of basis to complex space:
  $M : (l, \mathbb{R}) \rightarrow (l, \mathbb{C})$
  $\mathbb{R} \rightarrow M \times i: M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$

then:
  $O'(\phi) \equiv M^* O(\phi) M^{-1}$ [equiv. rep.]

- $O'(\phi)^{\times 2} = (e^{2\phi}, 0, 0, e^{-2\phi})$

- Note that $O'(\phi)$ has block-diagonal form.

- A series of $O'(\phi)$:
  $O'(\phi) = D^{(m)}(\phi) \oplus D^{(m)}(\phi)$

- Where $D^{(m)}(\phi) = e^{2m\phi}$ are irreps of $\text{SO}(2)$.

4.2. Generators of continuous (Lie) group

- **Definition**: Generators

- Generators are: fundamental objects of a Lie group, and are the matrices responsible for infinitesimal rotations.

- Taylor expansion of $O(\phi)$, $\phi \ll 1$:
  $\mathbb{R}$ about $\Phi = 0$:
  $\approx O(\phi)$ [equivalently, expand cosine and sine about $\phi = 0$.]

- $O'(\phi) = \begin{pmatrix} 1 - \frac{\phi^2}{2} + \frac{\phi^4}{4!} & \cdots & -\phi^2 + \frac{\phi^4}{3!} - \cdots \\ \frac{\phi^2}{2!} & \cdots & \frac{\phi^4}{4!} & \cdots \end{pmatrix}$

- $O(\phi) = O(0) + \frac{dO(\phi)}{d\phi}|_{\phi=0} + \frac{d^2O(\phi)}{d\phi^2}|_{\phi=0} O(\phi) + \cdots$

- $O(\phi) = O(0) + \frac{dO(\phi)}{d\phi}|_{\phi=0} + \frac{d^2O(\phi)}{d\phi^2}|_{\phi=0} O(\phi) + \cdots$
\[ O(\phi) = 1_2 + 5 \phi \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) + O(\phi^2) \]

\[ = 1_2 - \begin{array}{cc} 1 & \phi \\ \phi & 1 \end{array} \]

\[ \sigma_2 : \text{Pauli matrix} \]

From (2) we get:

\[ i \sigma_2 = \frac{d}{d\phi} O(\phi) \Big|_{\phi = 0} \]

\[ O(\phi) \equiv D^{(2)}_{SO(2)}(\phi) \]

- **Properties:** \( [\phi \equiv \phi \leq 2 \pi] \)

\[ O(\phi) = 1_2 - i\sigma_2 \phi \]

(i) \( O^\dagger O = 1 \Rightarrow \sigma_2 = \sigma_2^\dagger \)

(ii) \( \det(O(\phi)) = 1 \Rightarrow \text{Tr}(\sigma_2) - i \text{Tr}(\phi) = 1_2 \)

(iii) \( \sigma_2^2 = 1_2 \)

- For finite rotations, one can build up the exponential representation:

\[ O(\phi) = \lim_{N \to \infty} \left[ O(\phi/N) \right]^N \]
4.3. Role of Generators in QM.

A generator corresponds to a differential operator.

Consider a rotation about $z$-axis thru $\phi$:

$$\psi(y) = \psi(R^x)$$

$$\psi'(y) = \psi'(0)$$

Let:

$$\psi(y) = U_\phi \psi(0)$$

For $\phi \ll 1$:

$$U_\phi \equiv 1 + i \vec{\xi} \cdot \vec{r}$$

$$\psi(y) = (1 - i \vec{\xi} \cdot \vec{r}) \psi(0)$$

$$\psi'(y) = (\vec{\xi} \cdot \vec{r}) \psi(0)$$

Generator of rotation group:

$$\vec{\xi} = -i \frac{d}{dy}$$

Generator of $SO(2)$ corresponds to linear moment operator [Cartan] and orbital angular momentum operator [Poincaré] in $\mathbb{C}$.

In fact these correspond to translation and not rotation.

4.4. SO(3) Group

$SO(3)$: Group of all proper rotations in 3-dim about a given unit vector $\vec{n} = (n_x, n_y, n_z) = (n_1, n_2, n_3)$ with $R^2 = 1$.

Rotations about $x$, $y$, $z$-axes [or $x_1$, $x_2$, $x_3$-axes] are:

$$R_1(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{pmatrix}$$

$$R_2(\phi) = \begin{pmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{pmatrix}$$

$$R_3(\phi) = \begin{pmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The generators $\mathcal{X}$ of $SO(3)$ are:

$$-i \mathcal{X} = \frac{dR_2(\phi)}{d\phi} \bigg|_{\phi=0}$$

$$\mathcal{X}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\mathcal{X}_2 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{X}_3 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$

$$\{ \mathcal{X}_i \}$$

- General rep. of the group element of $SO(3)$:

$$R(\phi, \hat{n}) = e^{-i\phi \hat{n} \cdot \vec{r}}$$

$$\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3), \quad \hat{n} = (n_1, n_2, n_3)$$

4.4.1. $SO(3)$ Rotation about an arbitrary axis

Here we use the generators $\mathcal{X}_i$ to derive...
rep of \( \text{SO}(3) \) \[\text{rotation matrix}\] \( R(\phi, \mathbf{R}) \),
about any arbitrary axis. [\textit{i.e.} proof of (4)].

For rotation \( \phi \ll 1 \) about \( \mathbf{R} \),
\[
\mathbf{R} = (\cos \phi) \mathbf{I} + (\sin \phi) (\mathbf{R} \times \mathbf{I}) \]
\[
\mathbf{R} \rightarrow \mathbf{R}' = \mathbf{R} + \delta \mathbf{R} = \mathbf{L} + \phi (\mathbf{R} \times \mathbf{L})
\]
in components:
\[
R_i \rightarrow R'_i = R_i + \phi \epsilon_{ijk} \hat{n}_j R_k
\]
\[
= R_i - \phi \epsilon_{ijk} \hat{n}_j R_k
\]
\[
\leftrightarrow (R_{ij} - \phi \epsilon_{ijk} \hat{n}_k R_i) = (\mathbf{R} \times \hat{n}) R_i
\]
\[\text{after relabeling}
\]
Compare (2) & (3):
\[
8 \mathbf{R}_i = -\phi \epsilon_{ijk} \hat{n}_k R_j
\]
\[
= -i\phi (-i \epsilon_{ijk} \hat{n}_k) R_j
\]

Since:
\[
\mathbf{R}' = R(\phi, \mathbf{R}) \mathbf{R}
\]
\[
\mathbf{L}_3 - i\phi \hat{n}_3
\]
\[
\mathbf{L}_j
\]
Therefore:
\[
(X_{\hat{R}})_{ij} = -i \epsilon_{ijk} \hat{n}_k = \epsilon_{ijk} (X_L)_{ij}
\]
\[\text{eq.(4)}\]
i.e.,
\[
(X_{\hat{R}})_{ij} = (\hat{\mathbf{R}} \cdot \mathbf{X})_{ij}
\]
\[
\mathbf{X} = \langle X_1 X_2 \rangle
\]
\[\text{generator of a rotation about any axis} \]
\[\hat{\mathbf{R}}\text{ is:} \hat{\mathbf{R}} \cdot \mathbf{X} \text{ is generator of}
\]
\[\text{rotation about the fixed reference}
\]
\[\text{coordinate system.}\]

\[
\begin{align*}
\text{then a full rotation matrix about } \hat{\mathbf{R}} \text{ is}:
R(\phi, \hat{\mathbf{R}}) &= e^{i\phi \hat{\mathbf{R}} \
\text{SO}(3) } \\
\text{where:}
\det(R) &= 1 \Rightarrow \text{Tr} (\hat{\mathbf{R}} \cdot \mathbf{X}) = 0 : \text{traceless} \\
R^{-1} &= R^T \Rightarrow (\hat{\mathbf{R}} \cdot \mathbf{X}) = (\hat{\mathbf{R}} \cdot \mathbf{X})^T : \text{Hermitian}
\end{align*}
\]

4.4.2 Properties of the Generators of \( \text{SO}(3) \)

The generators, generally, form an Algebra [\textit{Lie Algebra}], or vector space, with two laws:

(i) Linear Combination:

LC of two generators produces a generator: \( \forall X_1, X_2 \in \text{GL}(N, \mathbb{C}) \):
\[X_1, X_2 \in \text{GL}(N, \mathbb{C})
\]
\[= X_3 \text{ : generator}
\]

(ii) Commutation:

the commutator, or Lie product, denoted [\ldots], of two generators is a generator:
\[\forall X_1, X_2 \in \text{GL}(N, \mathbb{C})
\]
\[X_1, X_2 \equiv X_1 X_2 - X_2 X_1 \in \text{GL}(N, \mathbb{C})
\]
\[= X_3 \text{ : generator}
\]
\[\forall\]

(iii) For \( \text{SO}(3) \): Commutation Relations:
\[
[X_i, X_j] = [X_i X_j - X_j X_i - i\epsilon_{ijk} X_k]
\]
\[\forall\]
Proof (E): 
Using 
\[ (X^i)_{ji} = -i \epsilon_{ijk} \]  
\[ \implies \epsilon_{ijm} \epsilon_{jnk} = \delta_{mk} \delta_{ij} - \delta_{ij} \delta_{mk} \]  
so \[ \text{compo } [x_i, x_j] = [x_i x_j - x_j x_i] \] 
\[ = (x_i)_{mk} (x_j)_{kl} - (x_j)_{mk} (x_i)_{kl} \]  
\[ = -\epsilon_{mink} \epsilon_{jkl} + \epsilon_{mink} \epsilon_{jlk} \]  
\[ = -\left[ \epsilon_{mik} \epsilon_{lj} - \epsilon_{lij} \epsilon_{mk} \right] + \left[ \epsilon_{mik} \epsilon_{lj} - \epsilon_{lij} \epsilon_{mk} \right] \]  
\[ = \epsilon_{lij} \epsilon_{mik} - \epsilon_{lij} \epsilon_{mik} \]  
\[ = \delta_{mj} \delta_{ik} - \delta_{ij} \delta_{mk} \]  
\[ = \epsilon_{lij} \epsilon_{mik} \]  
\[ = i \epsilon_{lij} (x_i)_{mk} \]  
\[ \implies [x_i, x_j] = i \epsilon_{ijk} x_k \]  
\text{G.F.} 

Note: \( \epsilon_{ijk} \) : Structure coefficient of SO(3)

Proof (F): 
- Let rotation about two axes \( \hat{\theta}, \hat{\gamma} \) 
- Related by rotation \( \theta \) about \( \hat{\gamma} \) 
\[ \hat{\gamma} = S(\theta) \hat{\theta} \]  
\[ S(\theta) = D(\theta) = R(\theta) \epsilon_{ijk} = e^{-i x_i \theta} \]  
- Rotation about \( \hat{\gamma} \) 
\[ R(\hat{\gamma}, \phi) = e^{-i X_{\hat{\gamma}} \phi} \]  
\[ \text{since } \hat{\theta} = (\pi, 0, 0) \implies \hat{\gamma} \cdot x = x \]  
\[ R(\hat{\theta}, \phi) = e^{-i X_{\hat{\theta}} \phi} \]  
- Rotation about \( \hat{\gamma}' \) is equivalent to rotation \( \hat{\gamma} \) 
\[ R(\hat{\gamma}', \phi) \sim R(\hat{\gamma}, \phi) \]  

4.4.3. Irreps of SO(3) 
- For SO(3) generators: 
- Jacobi Identity: 
\[ [x_{\alpha_1}, [x_{\alpha_2}, x_{\alpha_3}]] + [x_{\alpha_2}, [x_{\alpha_3}, x_{\alpha_1}]] + [x_{\alpha_1}, [x_{\alpha_2}, x_{\alpha_3}]] = 0 \]  
\[ = \left[ x_{\alpha_1}, [x_{\alpha_2}, x_{\alpha_3}] \right] = 0 \]  
where: 
\[ x_{\alpha_{ij}} = x_i x_j - x_j x_i = [x_i, x_j] \]  
\[ \epsilon_{(i} [\alpha_{ij}] = +1 \]  
- Need to find irreducible matrices which correspond to SO(3) generators: \( [X = (x_1, x_2, x_3)] \) 

1. satisfy Jacobi (1) and (2)
From QM, angular momenta operators $\widehat{\mathbf{S}}_x, \widehat{\mathbf{S}}_y, \widehat{\mathbf{S}}_z$ and satisfy the two properties:

$$\{\widehat{\mathbf{S}}_x, \widehat{\mathbf{S}}_y\} = i \hbar \widehat{\mathbf{S}}_z$$

$$[\widehat{J}_{\pm}(x, y)]: = 0$$

where $\hbar^2 = \sum_{i=1}^{3} \frac{\hbar^2}{2m_i}$ can be diagonalized with $\widehat{\mathbf{S}}_z$ in some basis \{\sinh [\frac{\hbar^2}{2m_i}] = 0\}\} one can know their eigenvalues $m$ and generate simultaneously.

Let:

$$\begin{align*}
\widehat{\mathbf{S}}_z |j, m\rangle &= j (\pm 1) |j, m\rangle \\
\frac{\hbar}{m} |j, m\rangle &= m |j, m\rangle
\end{align*}$$

for each $j$, there are $(2j+1)$ eigenvalues $|j, m\rangle$ and $(2j+1)$ eigenstates $|j, m\rangle$ of $\widehat{\mathbf{S}}_z$.

$\Rightarrow$ $\widehat{S}_z [\pm x, \pm x, \pm x]$ are $(2j+1) \times (2j+1)$-dim matrices with $(2j+1)$ eigenvalues and $(2j+1)$-dim basis.

\(\Rightarrow\) they are irreps of $SO(3)$.

Note: $\begin{cases} 
\widehat{X} = \widehat{\mathbf{S}} \\
\{ \widehat{X}_\delta = \widehat{\mathbf{S}}_\delta, \\
\widehat{X}_2 = \widehat{\mathbf{S}}_2, \\
\widehat{X}_1 = \widehat{\mathbf{S}}_1 \end{cases}$

matrix elements:

$$\begin{align*}
(X_3)_{nm} &= \langle j, m | \widehat{X}_3 | j, n \rangle = m \delta_{nm} \\
(X_\pm)_{nm} &= \langle j, m | \widehat{X}_\pm | j, n \rangle = \sqrt{j(j+1) - m(m \pm 1)} \delta_{nm, n' \pm 1}
\end{align*}$$

For $\widehat{X}_1$ and $\widehat{X}_2$, $[\widehat{X}_1, \widehat{X}_2, \widehat{X}_3]$ introduce ladder operators:

$$\widehat{X}_\pm = \widehat{X}_1 \pm i \hbar \widehat{X}_2$$

Example: $j = 1; m = \pm 1, 0$:

$$\langle j, m | \widehat{X}_3 | j, n \rangle = m \delta_{nm}$$

$$\begin{align*}
\widehat{X}_3 &= \frac{\hbar}{\kappa} \\
\kappa &= \pmatrix{ 1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0} \\
\phi &= \text{diag}(\cos \theta, \sin \theta, \sin \theta)
\end{align*}$$
4.5.1. SU(2) Group:

- Rotation of a complex 2-dim vector \( \mathbf{v} = (v_1, v_2) \), \( v_1, v_2 \in \mathbb{C} \), thru angle \( \theta \) about an axis \( [\hat{n}] \):

\[
\mathbf{v}' = U(\theta, \hat{n}) \mathbf{v}
\]

where:

- \( \det U = 1 \)
- \( U \mathbf{v} U^+ = \mathbf{v} \)

\[
U(\theta, \hat{n}) = e^{-i \hat{n} \cdot \mathbf{v} \theta / 2}
\]

- Generators of SU(2) are:
  \( \sigma_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \)

- Properties of \( \sigma_i \):
  1. \( \det U = 1 \Rightarrow \text{Tr}(\sigma_i) = 0 \)
  2. \( \mathbf{v} U U^+ = \mathbf{v} \Rightarrow \mathbf{v} = \mathbf{v}^+ \)
  3. \( \sigma_i \sigma_j + \sigma_j \sigma_i = 2 \varepsilon_{ijk} \sigma_k \)

- Commutation Relation:

\[
[X_i, X_j] = i \varepsilon_{ijk} X_k
\]

4.5. The SU(N) Group

- SU(N, \mathbb{C}) is the group of Special \[ \det U = 1 \]

Unitary \[ U^T = U^{-1} \] group of rotations of \( N \)-dim Complex vector thru any angle. It preserves the length of the vector:

\[
\sum_{i=1}^{N} |x_i|^2 = \sum_{i=1}^{N} |x_i'|^2
\]

\[ \text{Note: If we allow for half-integer values for } j \text{, we have different Algebra } \rightarrow \text{ SU}(2) \text{ Group, or Generally } \text{SU}(N); D(\mathfrak{su}(N)) \]

\[ \text{SO}(N) \text{ generators} \]

- \( \varepsilon_{ijk} \) Pauli matrices act on spin states:

\[
|1\rangle = |\frac{1}{2}, \frac{1}{2} \rangle, \quad |\downarrow\rangle = |\frac{1}{2}, -\frac{1}{2} \rangle
\]

which are, in fact, eigensates of \( \sigma_z \) \[ \sigma_z |\frac{1}{2}, \frac{1}{2} \rangle = \frac{1}{2} |\frac{1}{2}, 
\frac{1}{2} \rangle \]

Factor \( \frac{1}{2} \) corresponds to \[ m \neq \frac{1}{2} \] of \( \text{SO}(3) \) which we disallowed.
Relation between $SO(3)$ and $SU(2)$

For $SO(3)$:

$$R(\phi, \mathbf{A}) = e^{-i \mathbf{A} \cdot \mathbf{r} \cdot \mathbf{r}} = 1_3 - (\mathbf{A} \cdot \mathbf{r})^2 \left[ \cos \phi - i (\mathbf{A} \cdot \mathbf{r}) \sin \phi \right]$$

$$\Rightarrow \begin{cases} R(0, \mathbf{\hat{r}}) = 1_3 & \text{identity matrix of } SO(3) \\ R(2\pi, \mathbf{\hat{r}}) = -1_3 \end{cases}$$

For $SU(2)$:

$$U(\phi, \mathbf{A}) = e^{i \mathbf{A} \cdot \mathbf{r} \cdot \mathbf{r}} = 1_2 \cos \left( \frac{\phi}{2} \right) - i (\mathbf{A} \cdot \mathbf{r}) \sin \left( \frac{\phi}{2} \right)$$

$$\Rightarrow \begin{cases} U(0, \mathbf{\hat{r}}) = 1_2 \\ U(2\pi, \mathbf{\hat{r}}) = -1_2 \end{cases}$$

Consider a mapping from $SU(2)$ into $SO(3)$:

$$f : SU(2) \rightarrow SO(3)$$

Recall, kerf = \{ g \in SU(2) | f(g) = e_{SO(3)} = e_{SO(3)} \}

$$= \left\{ 1_2, -1_2 \right\}$$

Normal subgroup of $SU(2)$

$Z_2$ [Reflection group]

Theorem: Isomorphism Theorem

Let $(A, *)$ and $(B, *)$ be two groups and let $f$ be the mapping with kernel

$$f : A \rightarrow B$$

Then the quotient group, $A/\text{kerf}$, is isomorphic [1:1 correspondence] to group $B$:

$$A/\text{kerf} \cong B$$

This theorem implies

$$SO(3) \cong SU(2)/Z_2$$

4.5.2. Example: Isospin in $SU(2)$

$SU(2)$ group acts on isospin space, where states are "Spinors" called isospin or flavour state.

We can think of $u$ (up) and $d$ (down) quarks as isospin doublet.

Isospin

$$q = \begin{pmatrix} u \\ d \end{pmatrix}, \quad I_3(u) = +\frac{1}{2}, \quad I_3(d) = -\frac{1}{2}$$

Combining two such quarks $(u, d)$ is denoted as:

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

$1$ is isospin triplet: $I_3 = 1, 0, -1$

$0$ is singlet: $I_3 = 0$

$$\begin{align*}
I_3 &= 1 : \quad \pi^+ = u \bar{d} \\
I_3 &= -1 : \quad \pi^- = u \bar{d}
\end{align*}$$

$I_3 = 0 : \quad \pi^0 = \frac{1}{\sqrt{2}} (u \bar{u} - d \bar{d})$

$\pi$ can be written in terms of dimensions, corresponding spin degrees of freedom:

Each fermion has $1/2 = 2$. 

so 1 2 \otimes 1 2 = 1 \oplus 0
spin - dim: 2 \otimes 2 = 3 \oplus 1

Combining 3 quarks:

Isospin: 1 2 \otimes 1 2 \otimes 1 2 = (1 \oplus 0) \otimes 1 2 = 0 \oplus 1 2 \oplus 1 2
spin - dim: 2 \otimes 2 \otimes 2 = (3 \oplus 1) \otimes 2 = 4 \oplus 2 \oplus 2

One of the doublets is \( (P) \equiv (\text{proton}) \), since then there are more than two states, and they do not mix with other doublets. P and n can be related through a rotation in SU(2) isospin space. Therefore, expect p and n to have same mass. The symmetry [isospin rotation] is broken by both:

(a) Quark - constituent masses
(b) Electro-weak effects

5. **Tensors in SU(N)**

5.1. **Tensors in SU(N)**:

- Transformation of a complex vector

\[ \Psi_i = (\psi_i^1, \psi_i^2, \ldots, \psi_i^n) \] in SU(N):

\[ \Psi_i^j = U_{ij}^k \psi_j^k \]  \hspace{1cm} (5.1)

\[ U^T U = I = U U^T \quad \text{and} \quad \det U = 1 \]

- Define the scalar product invariant under SU(N):

\[ (\psi, \phi) \equiv \psi_i^* \phi_i = \psi_i^* \phi_j \] \hspace{1cm} (5.2)

Hence, the transform of the complex conjugate

\[ \Psi_i^* \rightarrow \Psi_i^{* j} = U_{j i}^{* k} \psi_i^k \]  \hspace{1cm} (5.3)

with

\[ U_{j i}^{* k} = U_{jk}^* \delta_{i j} \]

\[ U^* \left[ U_{ij} \right]^k = U^* U_{ik}^j = \delta^k_j \]

\[ U_{ij} \neq U_{ji} \]

- Higher rank tensors are defined as those quantities which have the same transform law as the direct (diagonal) product of vectors [and one-forms]:

\[ \Psi_{i_1 \ldots i_p} = U_{i_1 k_1}^{i_1} U_{i_2 k_2}^{i_2} \ldots U_{i_p k_p}^{i_p} \]

\[ (U_{i_1 k_1}^{i_1} U_{i_2 k_2}^{i_2} \ldots U_{i_p k_p}^{i_p})^T = U_{k_1 i_1}^{i_1} U_{k_2 i_2}^{i_2} \ldots U_{k_p i_p}^{i_p} \]

- The rank of \( \Psi_{i_1 \ldots i_p} \) is \( p+q \) with p contravariant and \( q \) covariant indices.

\[ \epsilon_{i_1 \ldots i_p}^{i_1 \ldots i_p} \]

- Kronecker delta \( \delta_{ij} \):

SU(N) transform of \( \delta_{ij} \):

\[ \delta^{ij} = U_{ik}^j \delta_{jk}^i \]

\[ \delta_{ij} = U^*_{ik} U_{jk}^i = \delta_{ij} \]

- Invariant under SU(N).

\[ \varepsilon_{i_1 \ldots i_p}^{i_1 \ldots i_p} \]

- Levi - Civita symbol \( \varepsilon_{i_1 \ldots i_p}^{i_1 \ldots i_p} \):

\[ \varepsilon_{i_1 \ldots i_p}^{i_1 \ldots i_p} = \begin{cases} 
1 & \text{if } (i_1, i_2, \ldots, i_p) \text{ is even permutation} \\
0 & \text{otherwise} \\
-1 & \text{if } (i_1, i_2, \ldots, i_p) \text{ is odd permutation} 
\end{cases} \]
6. Note that \( E_{i_1 i_2 \ldots i_n} \) is defined to be fully antisymmetric, such that
\[
E_{i_1 i_2 \ldots i_n} E^{i_1 i_2 \ldots i_n} = (n-1)! \delta^i_i
\]

- Invariance of \( E_{i_1 i_2 \ldots i_n} \) (and \( E^{i_1 i_2 \ldots i_n} \)) under
  SU(N) transf.: 
  \[
  E_{i_1 i_2 \ldots i_n} = U_{i_1}^j U_{i_2}^j \ldots U_{i_n}^j E_{j_1 j_2 \ldots j_n} 
  = \det U E_{i_1 i_2 \ldots i_n} 
  = E_{i_1 i_2 \ldots i_n}
  \]

* Reduction of higher rank tensors:
- Lower rank tensors can be formed by
  appropriate use of \( \delta^i_i \) and \( E^{i_1 i_2 \ldots i_n} \):
  \[
  \psi_{i_1 i_2 \ldots i_n} = \delta_{i_1}^{i_2} \delta_{i_2}^{i_3} \ldots \delta_{i_n}^{i_1} \psi_{i_3 i_4 \ldots i_n} 
  \]
  \[
  \psi_i = E_{i_1 i_2 \ldots i_n} \psi_{i_1 i_2 \ldots i_n} 
  \]
  \[
  \psi = E^{i_1 i_2 \ldots i_n} \psi_{i_1 i_2 \ldots i_n} 
  \]
  \[
  \psi_{i_1 i_2 \ldots i_n} = E_{i_1 i_2 \ldots i_n} \psi_{i_1 i_2 \ldots i_n}
  \]

- SU(2) transforms Complex vectors of 2-comp,
  \( \psi_a = (\psi_1, \psi_2) \), called Spinor:
  \[
  \psi_a = U_a^b \psi_b = U^{-1} \psi
  \]
  with
  \( U \in SU(2) : \det U = 1, \quad UU^* = I_2 \)

- Trans. of Complex. Conjugate (c.c.):
  \[
  \psi_a^* \equiv \psi_a^* 
  \]
  \[
  \psi^a = U^a_b \psi^b = U^{-1} \psi
  \]

- \( U^a_b \) is c.c. rep. of \( SU(2) \) since
  under mapping \( f: \quad U \rightarrow U^* \)
  \[
  U \rightarrow f(U) = U^*
  \]
  multiplication law is preserved:
  \[
  f(U_1 U_2) = (U_1 U_2)^* = U_2^* U_1^* = f(U_2) f(U_1)
  \]

- Equivalence of \( U \) and \( U^* \) in \( SU(2) \):
  Define
  \[
  C \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ; \quad C^2 = A_2^{(5,5)}
  \]
  recall,
  \[
  U(\theta, \varphi) = \begin{pmatrix} e^{i \frac{\varphi}{2}} & e^{i \frac{\theta}{2}} \\ -e^{-i \frac{\varphi}{2}} & e^{-i \frac{\theta}{2}} \end{pmatrix} 
  \]
  \[
  U(0, \varphi) = e^{i \frac{\varphi}{2}}
  \]

- then
  \[
  C^{\frac{1}{2}} C^{-1} = -C^* , \quad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
  \]

* Then
  \[
  C^{\frac{1}{2}} C^{-1} = C \begin{pmatrix} 1 \cos \frac{\varphi}{2} - i \sin \frac{\varphi}{2} \\ i \sin \frac{\varphi}{2} \cos \frac{\varphi}{2} \end{pmatrix}
  \]
  \[
  = \frac{1}{2} \cos \frac{\varphi}{2} + \frac{1}{2} (\sin \frac{\varphi}{2}) \sin \frac{\varphi}{2}
  \]
  \[
  = U^*
  \]

NB. Since the Levi-Civita tensor can be used to lower or raise indices, we only
need to study tensors with lower indices.

Eg. \( SU(N) \) have
\[
\psi_i = U_{i_1}^j U_{i_2}^j \ldots U_{i_n}^j \psi_{j_1 j_2 \ldots j_n}
\]
\[
U_{i_1}^j U_{i_2}^j \ldots U_{i_n}^j = \delta_{i_1}^i \delta_{i_2}^j \ldots \delta_{i_n}^k
\]
\[
= \psi_i : \text{ invariant.}
\]
From (8.3): $C^a_b = \varepsilon^{ab}$: Levi-Civita tensor.

- $C$ can be used [since $\sigma_\mu$ is just $\varepsilon^{\mu\nu}$] to raise or lower indices:
  \[ \psi_a = C_{ab}^c \psi^b \]

- Properties of $C (\varepsilon)$ matrix:
  1. **SU(2) invariance**: i.e.,
     \[ UC/U^T = C \]
     \[ \Leftrightarrow C_{ab} = U_a^c U_b^d C_{cd} = U_{ac} U_{bd} C_{cd} \]

     with $c^2 = -1_2$, det $C = 1$.

     **Proof:**
     \[ c^2 = -1_2 \Rightarrow UC = -U1_2 = -2U \]
     \[ \Rightarrow C = C^2 U \]
     \[ \Rightarrow c^T U c = c^T C c = c^T U C \]
     \[ \Rightarrow C^T UC = C UC = U^* \]

     Hence:
     \[ U^* = (U^T)^T = (U^{-1})^T ; \ U^T = U^{-1} \] in SU(2)

     \[ \Rightarrow UC = C (U^{-1})^T \]
     \[ \Rightarrow \begin{bmatrix} UC & U^T \end{bmatrix} = C \]
     \[ \Rightarrow \begin{bmatrix} U^T \end{bmatrix} UC = C \]

- **Note:**
  \[ \psi_a = C_{ab} \psi^b \Rightarrow \varepsilon^{ab} \psi^a \]

  $E_{ab}$ is analogous to $\eta_{ab}$ in Special Relativity.

  where 4-tensors:
  \[ A_a = \eta_{ab} A^b \quad a = 0, 1, 2 \]

  where $\eta_{ab} = \delta_{ab}$ also $E_{ab} = \varepsilon_{ab}$

  **Scalar product of spinors**
  \[ (\phi, \psi) = \phi^a \varepsilon^{ab} \psi^b \]

  It is invariant under SU(2).

  **Proof:**
  \[ \psi_a \phi^a \Rightarrow \phi^a \phi_a = \varepsilon^{ab} \psi^a \psi^b \]
  \[ = \varepsilon^{ab} (U_{ac} \psi^c) (U_{bd} \phi_a) \]
  \[ = (\varepsilon U)^{a_d} \psi_{cd} \phi_a \]
  \[ = (U^T \varepsilon U)^{cd} \psi_{cd} \phi_a \]
  \[ = \varepsilon^{cd} \psi_{cd} \phi_a = \varepsilon^{cd} \phi^c = \varepsilon^{cd} \phi^c \cdot \]

  **5.2.1, SU(2): Constructing higher dim. irreps and Clebsch-Gordan Series**

  - Combining two 2-dim spinors gives 4-dim spinor:
    \[ \gamma_{ab} = \gamma_{a} \phi_{b} \]

  - To write irreps, which transform $\gamma_{ab}$, we
    
    **Symmetric and antisymmetric:**
    \[ \gamma_{ab} = \frac{1}{2} \left[ \gamma_{(ab)} + \gamma_{[ab]} \right] \]

    \[ \Rightarrow \left\{ \begin{array}{l}
    \gamma_{a+b} : \text{has 3 comp.} \\
    \gamma_{a-b} : \text{1 comp} = \gamma_{a} \phi_{a} - \gamma_{b} \phi_{b} \\
    \end{array} \right. \]
    \[ = \varepsilon_{ab} \gamma_{a} \phi_{b} = \varepsilon_{ab} \gamma_{a} \phi_{a} \]

    i.e., scalar under SU(2) \[ \varepsilon_{ab} \] is "metric" of SU(2)

  **Theorem:**
  Symmetric and antisymmetric parts of a rank-2 tensor transform independently, i.e., they form invariant subspaces.
In terms of Young Tableaux (next section), this process is represented as:

\[ \nu_0 = [2], \quad \text{dim} = 2 \quad [SU(2)] \]

Combining \( \nu_0 \times \nu_0 = [1] \times [1] \)

Symmetry: \( [1] \)

Antisymmetry: \( \frac{[1]}{\sqrt{2}} \)

\( \square \) is 1-dim scalar

\( \Box \) is 3-dim irrep of 4D combined

\( \square \times \square = \Box + \square \)

This process represents how a combined system of spinors [basis of SU(2)] transform under SU(2) rotation.

\[ \text{Spin Correspondence: } (x \rightarrow \Theta) \]

\[ \frac{1}{2} \oplus \frac{1}{2} \oplus 0 = \frac{3}{2} \]

\[ (1 \oplus 0) \times \frac{1}{2} = \frac{3}{2} \oplus \frac{1}{2} \]

\[ \text{dim} = 4 \quad \text{dim} = 2 \]

Note spinor \( \psi = (\psi_{\frac{1}{2}}, \psi_{-\frac{1}{2}}) \)

5.2.1. Irreps and dimension in SU(2)

Generally, irreps are carried by tensors with \( n \)-symmetric indices \( \phi_{(a_1 \ldots a_n)} \).

For SU(2): \( \text{dim} [\phi_{(a_1 \ldots a_n)}] = n+1 \)

When we multiply too many tensors, we obtain higher dimension rep. Which is then reducible:

\[ \text{dim} = \begin{pmatrix} n+m+3 \\ n \times m \\ (n+m) \times \square \end{pmatrix} \quad \text{in SU(2)} \]
One by one antisymmetrizing causes to loose $2 \left[ \text{in } SU(2) \right] = 2$ indices $2$, so 2 dimensions resulting in:

$$\sum_{\text{boxes}} \sum_{\text{boxes}} = \sum_{i=0}^{n+m} \sum_{\text{boxes}}$$

where sum increments in steps of 2 [no for $SU(2)$].

$$4 \times 3 = 2 \oplus 4 \oplus 6$$

If denote:

$$\begin{cases} m = 2j_1 \\ n = 2j_2 \end{cases}$$

then

$$\left( 2j + 1 \right) \left( \frac{1}{2} \right) \sum_{j=0}^{j_m} \sum_{j=0}^{j_n}$$

in terms of dimension.

...this is the rule of adding spins, [or total angular momenta].

5.3. $SU(3)$ Tensors

$SU(3)$ basis is a 3 comp. vector [in relation to Particle Physics], with 36 comp. representing Colour [red $\times r$, blue $\times b$, green $\times g$] state of a quark, :

$$\psi_a = \begin{pmatrix} f \\ b \\ c \end{pmatrix}$$

$$\psi^a = \begin{pmatrix} \bar{f} \\ \bar{b} \\ \bar{c} \end{pmatrix}$$

where:

$$\psi_a = 3, \quad \psi^a = \bar{3}, \quad \psi^{\ast a} = 3$$

*Complex. conj.*

Under $SU(3)$, it transforms as:

$$\begin{pmatrix} \psi_a \\ \psi^a \end{pmatrix} = \begin{pmatrix} \bar{U}^a_b \psi_b \\ \bar{U}^{\ast a}_b \psi^b \end{pmatrix}$$

Note the complex conj. rep, $\psi^a$, corresponds to rational of antiparticle, $\psi^a$, is not equivalent to $\psi^a$, i.e.

there is not matrix such that:

$$U C U^\dagger = U$$

equivalently:

$$\psi_a \neq C_{ab} \psi^b$$

But, instead:

$$\psi_a = E_{abc} \psi^{[bc]}$$

i.e. basis transform as a pair of antiparticle.

Lower indice, where $E_{abc}$ is invariant under $SU(3)$; quark $\equiv \psi_a$ quark $\equiv \psi^a$.

Also can form scalar invariants, as before, by contraction:

$$\psi_0 \psi^0 = \psi_a E_{abc} \psi^{[bc]}$$

$\psi$ skill can decompose multi-index tensor into irreps. by symt/anti-symt. and contracting with $E$.

Examples:

(i) Quark system under $SU(3)$: $q \equiv \psi = 3, \bar{q} = \bar{3}$

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \end{pmatrix} = \begin{pmatrix} 6 \oplus \bar{3} \oplus \bar{3} \oplus 6 \end{pmatrix}$$

- Totally anti-sym. $E_{[abc]}$

- Singlet = boson $\rightarrow$ meson

(ii) $qq'$ system.

$$\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 6 \oplus \bar{3} \oplus \bar{3} \oplus 6 \end{pmatrix}$$

(iii) $qqq$ (baryon):

$$\begin{pmatrix} 3 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \end{pmatrix} = \begin{pmatrix} 10 \oplus 8 \oplus 8 \oplus 1 \end{pmatrix}$$

*Baryon*
5.4. SU(N) & Young Tableaux

5.4.1. Permutation Operator $P_{ij} \in S_n$

Higher-rank SU(N) tensors, $\phi_{\alpha_1 \alpha_2 \ldots \alpha_N}$, do not generally define bases of irreps. To decompose them into bases of irreps, we exploit the following property [which is at the heart of Young Tableaux]:

Consider the 2nd rank tensor, $\chi_{ij}$, with the transfer property:

$$\chi'_{ij} = U_i^k U_j^l \chi_{kl}$$

Permutations of $i \rightarrow j$ denoted by $P_{ij}$ do not change the transfer law:

$$P_{ij} \chi'_{ij} = \chi'_{ij} = U_i^k U_j^l \chi_{kl} = \chi_{ij}$$

Let:

$$U_i^k = U_i^k U_j^l \chi_{kl}$$

$$P_{ij} = U_i^k P_{ij} U_j^l$$

**NB:** If we treat $P_{ij}$ as an operator, such that

$$P_{ij} \left[ A_{ij} B_{kl} \right] = \left[ P_{ij} A_{ij} \right] B_{kl} + P_{ij} \left[ B_{kl} \right]$$

then we find:

$$\left\{ U_i^k U_j^l \right\} = \left( U_i^k U_j^l \right)^* P_{ij}$$

with:

$$U_i^k = U_i^k \otimes \left( A \otimes B \right)^*$$

Hence $P_{ij}$ can be used to construct the following irreps: [Formally true: irreducible basis]

$$S_{ij} = \frac{1}{2} (1 + P_{ij}) \chi_{ij} = \frac{1}{2} (\chi_{ij} + \chi_{ji}) \equiv \frac{1}{2} \chi_{ij}$$

$$A_{ij} = \frac{1}{2} (1 - P_{ij}) \chi_{ij} = \frac{1}{2} (\chi_{ij} - \chi_{ji}) \equiv \frac{1}{2} \chi_{ij}$$

with:

$$P_{ij} S_{ij} = S_{ij}$$

$$P_{ij} A_{ij} = -A_{ij}$$

Since there is no mixing between $S_{ij}$ and $A_{ij}$ under SU(N) transf.

$$S_{ij} = \frac{1}{2} \left( \chi_{ij} + \chi_{ji} \right)$$

$$A_{ij} = U_i^k U_j^l \chi_{kl}$$

5.4.2. Young Tableaux

- A complex (covariant) vector (or 2-tensor) $\chi_{ij}$ in SU(N) is represented as

$$\chi_{ij} \equiv \left[ \begin{array}{c} \chi_{ij} \\ \chi_{ji} \end{array} \right]$$

- The operation of symmetrization and antisymmetrization is represented as:

$$\psi_{\left(\begin{array}{c} i \\ j \end{array}\right)} \equiv \left[ \begin{array}{c} \chi_{ij} \\ \chi_{ji} \end{array} \right]$$

with:

$$\psi_{\left(\begin{array}{c} i \\ j \end{array}\right)} = \frac{1}{2} \left( 1 + P_{ij} \right) \chi_{ij} = \chi_{ij} = S_{ij}$$

$$\psi_{\left[\begin{array}{c} i \\ j \end{array}\right]} = \frac{1}{2} \left( 1 - P_{ij} \right) \chi_{ij} = -A_{ij}$$

- By analogy, For $\psi_{\left(\begin{array}{c} i \\ j \end{array}\right) k}$ we have

$$\psi_{\left(\begin{array}{c} i \\ j \end{array}\right) k} \equiv \left[ \begin{array}{c} \chi_{ijk} \\ \chi_{ikj} \end{array} \right]$$

$$\psi_{\left[\begin{array}{c} i \\ j \end{array}\right) k} \equiv \left[ \begin{array}{c} \chi_{ijk} \\ -\chi_{ikj} \end{array} \right]$$

$$\psi_{\left(\begin{array}{c} i \\ j \end{array}\right) k} \equiv \left[ \begin{array}{c} \chi_{ijk} \\ \chi_{ikj} \end{array} \right]$$

$$\psi_{\left[\begin{array}{c} i \\ j \end{array}\right) k} \equiv \left[ \begin{array}{c} \chi_{ijk} \\ -\chi_{ikj} \end{array} \right]$$
where \( \psi_{ik;i} \) is totally symmetric in \( i, j, k \).
\( \psi_{i;j;k} \) is totally antisymmetric in \( i, j, k \).

\[
\psi_{i;j;k} = (1 - P_{ik}) (1 + P_{ik}) \psi_{i;j;k}
\]

\[
= (1 - P_{ik}) \left[ \psi_{i;j,k} + \psi_{i,j;k} \right]
\]

\[
= \psi_{i;j,k} - \psi_{i,j;k} - \psi_{i,j;k}
\]

**Rules for Constructing a legal Young Tableaux**

- A typical Young Tableaux for an \( (n \text{-rank}) \) tensor with \( n \) indices looks like:

  ![Example Young Tableaux](example.png)

- Each row of a Young Tableaux must contain no more boxes than the row above it.

  - Example: ![Not a valid diagram](not_valid.png)

- There should be no column with more than \( N \) boxes. A column with exactly \( N \) boxes can be crossed out.

  - Example: in \( SU(3) \):

    ![Example](example_su3.png)

**How to Find the dimension of a Young Tableaux**

1. Write down the ratio of two copies of the tableaux:

   - \( \text{SU}(3) \):

     \[
     \frac{1}{3} \quad \frac{3}{1} \quad \frac{3}{1}
     \]

   - \( \text{SU}(3) \):

     \[
     \frac{1}{3} \quad \frac{3}{1} \quad \frac{3}{1}
     \]

2. Numerator (Num): Start with the number \( N \) for \( SU(n) \) in the top left box. Each increase by \( (n-1) \) for each successive column [moving to the right in a row], and decrease by \( 1 \) for each successive row [moving down in the column].

   ![Numerator Diagram](numerator.png)

   - \( \text{Num} = \text{product of all entries} \)

3. Denominator (Den): In each box, write the number of boxes being to its right + the number of boxes being below it and add 1 for itself.

   ![Denominator Diagram](denominator.png)

   - \( \text{Den} = \text{product of all entries} \)

4. The dimension \( d \) of the rep is:

   \[
   d = \frac{\text{Num}}{\text{Den}}
   \]

   \[
   d = \frac{(N+1)(N+2)(N-1)N(N-2)(N-3)}{6 \times 4 \times 1 \times 4 \times 2 \times 3 \times 1 \times 1}
   \]

**Example:**

- \( SU(3) \):

  - \( \frac{1}{3} \): \( 3 \times 2 = 6 \)

  - \( \frac{3}{1} \): \( 3 \times 2 = 6 \)

  - \( \frac{3}{1} \): \( 3 \times 2 = 6 \)

- \( SU(3) \):

  - \( \frac{1}{3} \): \( 3 \times 2 = 6 \)

  - \( \frac{3}{1} \): \( 3 \times 2 = 6 \)

  - \( \frac{3}{1} \): \( 3 \times 2 = 6 \)

- \( \frac{1}{3} \): \( 3 \times 2 = 6 \)

- \( \frac{3}{1} \): \( 3 \times 2 = 6 \)

- \( \frac{3}{1} \): \( 3 \times 2 = 6 \)
Rules for Clebsch-Gordan Series

1. The direct product of irreps can be decomposed as a Clebsch-Gordan series (or direct sum) of irreps. This reduction can be performed systematically by means of Young Tableaux following the rules below:

(a) Write down the two tableaux $T_1$ and $T_2$ and label successive rows of $T_2$ with indices $a, b, c, \ldots$

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

(b) Attach boxes $a, b, c, \ldots$ from $T_2$ to $T_1$ in all possible ways one at a time. The resulting diagram should be:
- a legal Young Tableaux
- contain no two $a's$ or $b's$ or $c's$ in the same column [because of cancellation due to antisymmetrisation]

(c) At any given box position, there should be no more $b's$ than $a's$ to the right and above it. Likewise, there should be no more $c's$ than $b's$ etc.

(d) If there are multiple tableaux of the same shape, one keeps all except those where the labeling is identical. Whence one copy is kept

(e) Exclude (remove) columns with $N$ boxes in SU(N) as they correspond to scalars.

Example in SU(3):

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\times
\begin{array}{|c|c|c|}
\hline
a & a & a \\
\hline
b & b & c \\
\hline
\end{array}
\]
(F) Mesons:

- Clebsch-Gordan series:

\[
\begin{aligned}
3 \otimes 3 & = 8 \oplus 1, \\
\frac{\mathbf{q}_1 \mathbf{q}_2}{SU(N)} & = N^2 - 1, \\
\text{For } SU(3), \quad \mathbf{\omega}_{	ext{fin.}} & = 8.
\end{aligned}
\]

- Singlet plane is:

\[
\eta_4 = \frac{1}{\sqrt{3}} \mathbf{q}_1 \mathbf{q}_2 = \frac{1}{\sqrt{3}} (u\overline{u} + d\overline{d} + s\overline{s})
\]

- The remaining 8 components represent the pseudoscalar octet:

\[
\mathbf{P}^i = \left( \mathbf{q}_2 q^i - \frac{1}{3} \delta^i_j \mathbf{q}_1 \mathbf{q}_1 \right)
\]

- Where:

\[
\begin{aligned}
\mathbf{P}^+ & = u\overline{d}, \\
\mathbf{P}^- & = u\overline{u} - d\overline{d} - s\overline{s}, \\
\mathbf{K}^+ & = u\overline{d}, \\
\mathbf{K}^- & = u\overline{u} - d\overline{d} - s\overline{s}.
\end{aligned}
\]

(ii) Baryons as three-quark state:

- C.G. Series:

\[
3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus \overline{10} \oplus 1
\]

- Define:

\[
q_{ijk} = q_i q_j q_k
\]

- For example, the baryon octet may be represented by

\[
\begin{pmatrix}
1 & 1 & 1 \\
1 & -1 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

5.4.3. Applications to Particle Physics [Revised]

(A) The SU(3) Quark Symmetry

- Define the quark basis states:

\[
q_i = \begin{pmatrix} u \\ d \end{pmatrix}, \quad q^i = \begin{pmatrix} \overline{d} \\ \overline{u} \end{pmatrix}
\]

with \( q_i^2 = 3 \) and \( q^i = \overline{q}^i = \overline{3} \).
5.5. Lie Algebra

5.5.1. Generators of a group as Basis

Vectors of a Lie Algebra.

Definition: Lie Algebra

A Lie Algebra, \( L \), is defined by a set of a number \( d(G) \) of generators \( T_a \) closed under commutation:

\[
[T_a, T_b] = i f_{ab}^c T_c
\]

where \( f_{ab}^c \) are called structure constants of \( L \).

In addition, the generators \( T_a \) satisfy the Jacobi identity:

\[
[T_a, [T_b, T_c]] + [T_b, [T_c, T_a]] + [T_c, [T_a, T_b]] = 0.
\]

The set \( \{ T_a \} \) of generators define a basis of a \( d(G) \)-dimensional Vector Space \( (V, i) \).

In the fundamental rep. \( T_a \) are represented by \( d(F) \times d(F) \) matrices, where \( d(F) \) is the least number of dimensions needed to generate the continuous group. [i.e., for \( SU(N), d(F) = N \).]

Example:

(i) \( SO(3) \): \( T_a = X_a \)

(ii) \( U(1) \): \( T_a = 1 \)

(iii) \( SU(2) \): \( T_a = \frac{1}{2} \sigma_a \)

(iv) \( SU(3) \): \( T_a = \frac{2}{3} \epsilon_{ab} \)

(v) \( SU(3) \): \( T_a = \frac{2}{3} \epsilon_{ab} \)
Exponentiation of $T_a$ generates the group elements [rotation matrix] of the group. Continuous Lie group:
\[ R(0, \alpha) = e^{i \alpha \vec{R} \cdot \vec{T}} \]

with:
\[
\begin{align*}
\mathbb{T} &= \{ T_a \}_{a=1}^{d(\mathfrak{g})} = (T_1, T_2, \ldots, T_d(\mathfrak{g})) \\
\vec{n} &= (n_1, \ldots, n_d(\mathfrak{g})) & \vec{n} \cdot \vec{n} &= 1
\end{align*}
\]

\[ \text{NB:} \]
\begin{align*}
\text{Lie Algebra (L)} & \quad \text{GR Tensors Algebra (G)} \\
\begin{array}{c|c}
\text{Matrix} & \text{Vector} \\
T_a & \varepsilon_a \\
\hline
[T_a, T_b] = i f_{a b}^c T_c & \varepsilon_b \varepsilon_a \\
(T_a, T_b) = g_{a b} & \varepsilon_b \varepsilon_a \\
[D_\lambda (T_a)]^b = i f_{a b}^c & (\varepsilon_b \varepsilon_a) \\
g_{a b} = -f_{a c} f_{b d} & = -f_{a c} f_{b d}
\end{array}
\end{align*}

For every given $T_a \in L$, $[T_a, ]$ may be represented in the vector space $L$ by the structure constants themselves:
\[ [D_\lambda (T_a)]^c_b = i f_{a b}^c (\varepsilon_b \varepsilon_a) \]

Such a rep. of $T_a$ is called the Adjoint representation, denoted by $\mathcal{A}$.

The Killing form is defined as:
\[ g_{a b} = (T_a, T_b) = \text{Tr} [D_\lambda (T_a) D_\lambda (T_b)] = \text{Tr} [D_\lambda (T_a) D_\lambda (T_b)] \\
g_{a b} = -f_{a c} f_{b d} \quad \text{Cartan metric}
\]

The Cartan metric can be used to lower the index of $f_{a b}^c$:
\[ f_{a b c} = g_{a b} f_{b c}^d \\
\text{note:} \\
f_{c a b} = g_{c d} f_{d b}^a
\]

\[ f_{c a b} = \text{Tr} [D_\lambda (T_a) D_\lambda (T_b)] = \text{Tr} [D_\lambda (T_a) D_\lambda (T_b)] = \text{Tr} [D_\lambda (T_a) D_\lambda (T_b)] = -2 \text{Tr} [D_\lambda (T_a) D_\lambda (T_b)] \\
\]

\section{The Adjoint Representation}

The Lie Algebra's commutator $[T_a, ]$ (for fixed $T_c$) defines a linear homomorphism from $L$ to $L$ over $\mathbb{C}$:
\[ [T_c, ] : L \rightarrow L \\
[T_c, \lambda_1 T_a + \lambda_2 T_b] = \lambda_1 [T_c, T_a] + \lambda_2 [T_c, T_b] \\
\forall T_a, T_b \in L, \forall \lambda_1, \lambda_2 \in \mathbb{C}.\]
5.5.3. Normalization of Generators & Casimir Operator

- The generators of a Lie group $D_R(T_a)$ of a given rep. $R$ are normalized as:

$$\text{Tr} \left[ D_R(T_a) D_R(T_b) \right] = T_R \delta_{ab}$$

**Example:**

For $SU(N)$ (or $SO(N)$)

$T_R = T = \frac{1}{2}$, fundamental rep.

$T_A = N >$, Adjoint rep.

**Casimir Operator:**

Casimir operators $T_R^2$ of a Lie algebra of a rep $R$ are matrix reps that commute with all generators of $L$ in rep $R$:

$$\left[ T_R^2, T_R \right] = 0$$

- A construction of a Casimir Operator $T_R^2$ in a given rep $R$ of $SU(N)$ (or $SO(N)$) may be obtained by

$$\left( T_R^2 \right)_{ij} = T_R \sum_{\alpha, b=1} \sum_{k=1} \left[ D_R(T_a) \right]_{ik} g^{ab} \left[ D_R(T_b) \right]_{kj}$$

where $g^{ab} g_{bc} = \delta^a_c$; inverse Cartan metric

- In terms of generators $T_a$, Casimir Operator reduces to:

$$T_R^2 = \sum_a T_a T_a$$

**Schur’s Lemma:**

- $T^2$ is unit matrix

[Formally: $T_R^2 = T_R^{RC}$, so in (**)]

**Example:**

An example of Casimir Operators in angular momt. Oper. in QM:

$$J^2 = J_x^2 + J_y^2 + J_z^2$$

with $[J^2, J_z] = 0$, $z = x, y, z$.

---

6. Symmetry of the Lagrangian in QFT

- The Lagrangian density and principle of least action reside at the heart of modern QFT.

- There are two types of symt:

  (i) Global Symmetry:

  Group generators are independent of space/time points (events) i.e., constant phase rotation of fields give rise to conservative laws.

  (ii) Local (Gauge) Symmetry:

  Generators depend on specific phase rotations. i.e., local phase transformation give rise to force (interaction).

**NB:**

Is it possible so far, no reason is known for symt, Gauge symt $\leftrightarrow$ Force.
Given a gauge symmetry, one can compute the properties of the gauge symmetries. The first example is the gauge force [EM, weak, strong]. A second example is gravity force.

6.1. Maxwell's equations:

- Maxwell's equations in space with sources [E, J]:
  \[ \nabla \cdot E = \varepsilon_0 \varepsilon_0 \]
  \[ \nabla \times E = -\frac{\partial B}{\partial t} \]
  \[ \nabla \cdot B = 0 \]
  \[ \nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \]

- Introduce potentials:
  \[ B = \nabla \times A, \quad E = -\nabla \phi - \frac{\partial A}{\partial t} \]

There are not unique, we can define new potentials that satisfy some property on \( A \) and \( \phi \):

\[ \begin{cases} A &\rightarrow \tilde{A} = A + \nabla \phi' \\ \phi &\rightarrow \tilde{\phi} = \phi - \frac{\partial \phi}{\partial t} \end{cases} \]

These are called gauge transformations, which leave the physical quantities (only observed) unchanged, since \( A \) and \( \phi \) are scalar functions.

6.2. Covariant formulation:

- Introduce \( A^\mu = (E^\mu, A^\mu) \): 4-potential
- \( J^\mu = (\rho, J^\mu) \): 4-current density
- \( \partial^\mu = \left( \frac{1}{c^2} \frac{\partial}{\partial t}, \nabla \right) \): 4-differential operator.

Wave equa. (6.3) gives:

\[ \partial_\mu A^\mu = 0 \]

Lorentz gauge (6.16):

\[ \partial_\mu A^\mu = 0 \]

Tensor:

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \]

where:

\[ E_{\nu} = \frac{\partial A_\nu}{\partial \tau}, \quad B_{\nu} = \frac{1}{2} \varepsilon_{\nu\alpha\beta\gamma} F_{\alpha\beta} \]

6.11. Gauge Choice and wave. eqn.

Standard procedure:

\[ \frac{\nabla \times (\nabla \times A)}{\nabla \times (\nabla \times A) - \nabla^2 A} \]

Suppose we start off with potentials which do not satisfy this gauge condition. A transf. to new potential that do, under some condition on gauge function \( \gamma \):

Using (1):

\[ \nabla \cdot \frac{\partial E}{\partial \tau} + \nabla \cdot \nabla A = 0 \]

Provided \( \gamma \) satisfies (6.2), we find:

\[ \begin{cases} \nabla \cdot \nabla A = \phi \text{ or } \phi \text{ or } \phi = \varepsilon_0 \end{cases} \]

6.1. Maxwells equations:
6.2. Non-Relativistic QM

Schrödinger equ. (SE) for a particle \( m/e \) in electrostatic potential \( \phi \)

\[
i \hbar \frac{\partial \psi}{\partial t} = \left( \frac{-\hbar^2}{2m} \nabla^2 + e\phi \right) \psi
\]

Potential \( \phi \) and wavefunct \( \psi \) are unphysic (not observed). The physical quantities are:

(i) prob. density \( \zeta = \psi^* \psi \)

(ii) operat. expect. value \( \langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle \)

(iii) prob. Current \( \mathcal{J} = (\nabla \psi)^* \psi - \psi \nabla \psi^* \)

For boson

Under phase transf.

\( \psi \rightarrow \psi' = e^{i\alpha} \psi \)

they are invariant \( \text{[when } \alpha(x), \text{ gauge transf.]} \)

of \( \phi, A \) will cancel out the non-invariant term

\( \psi \rightarrow \psi' \text{ not invariant]}. \)

(2) Global phase : \( \alpha = \text{constant} \)

\[
i \hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi
\]

SE invariant

(3) Gauge Phase : \( \alpha = \alpha(x) \)

\[
i \hbar \frac{\partial \psi}{\partial t} + \frac{\hbar e}{2m} \frac{\partial}{\partial x} = \left( \frac{-\hbar^2}{2m} \nabla^2 + e\phi \right) \psi
\]

redefining or making gauge transf.

\( \psi \rightarrow \psi' = e^{-i\frac{\hbar}{m} \frac{2\phi}{e}} \psi \)

\[
i \hbar \frac{\partial \psi}{\partial t} = \left( \frac{-\hbar^2}{2m} \nabla^2 + e\phi \right) \psi
\]

For SE to be left invariant, we must introduce vector potential \( A \) via Minimal Substitution

\[
-i \hbar \nabla \psi \rightarrow \left( i \hbar \nabla - \frac{e}{c} A \right) \psi
\]

such that \( A \) is gauge transf. simultaneously as phase transf:

\( A \rightarrow A' = A + \frac{e}{c} \nabla \alpha \)

\[
i \hbar \frac{\partial \psi}{\partial t} = \left[ \frac{-\hbar^2}{2m} \nabla^2 + (\nabla - \frac{e}{c} \frac{\partial A}{\partial x})^2 + e\phi \right] \psi
\]

which is invariant.

i.e., there is a connection between gauge invariance and interactions. Here \( e \psi \phi \) produces EM interaction.

Such that inter. compensate for the gauge freedom of the phase [like with Maxwell: equ. 9 Lorentz gauge]

this is why. this phase transf. is called gauge transf.
6.3. Quantum Fields & Interaction

GFT = QM + SR (Relativity)

Lagrangian:
\[ L = \int d^4x \left( \frac{1}{2} \left( \partial \phi(x) \right)^2 - \frac{1}{2} m^2 \phi^2(x) \right) \]

Equivalence: \( \phi(x) \sim x \) in classical
\( 2 \phi(x) \sim \mathbf{p} \) (momentum).

The fundamental principle that tells us what path systems follow is the action principle:
\[ \frac{\delta}{\delta \phi(x)} \int dt \left[ L \right] = 0 \]

\( \Rightarrow \) EOM of QM. Field

\[ \left[ \frac{\partial \phi}{\partial x^\mu} - \frac{\partial L}{\partial (\partial \phi/\partial x^\mu)} \right] = 0 \]

EOM from E-L equs:
\[ \partial_\nu F_{\mu \nu} = \partial_\lambda A_\nu - \partial_\nu A_\lambda = -J^\lambda \]

with Lorentz gauge:
\[ \partial_\nu F_{\mu \nu} = -\partial_\lambda A_\nu - \partial_\nu A_\lambda = -J^\lambda \]

To remove the 2 unphysical dof, we introduce gauge-fixing term in Lagrangian:
\[ L_{GF} = \frac{1}{2 \xi} \left( \partial_\mu A_\nu \right)^2 \]

6.3.2. Vector (Photon) Field:

Lagrangian for Maxwell's equs:
\[ L_{ME} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - J_\mu A^\mu \]

6.3.3. Spinor (Fermion) Fields:

Lagrangian of Dirac equs:
\[ L_D = \bar{\Psi} \left( i \gamma^\mu \partial_\mu - m \right) \Psi \]

where:
\[ \Psi(x) = \left( \begin{array}{c} \Phi_1(x) \\ \Phi_2(x) \end{array} \right), \quad \bar{\Psi} = \left( \begin{array}{c} \Phi_1(x) \\ \Phi_2(x) \end{array} \right) \]
\[
\gamma^\mu = \begin{pmatrix}
0 \\
(\bar{\sigma}^\mu)^\gamma_eta \\
(\bar{\sigma}^\mu)^\gamma_i \\
0
\end{pmatrix}
\]
\[
\sigma^\mu = \begin{pmatrix}
1_2 \\
\vec{\sigma} \\
\vec{\sigma}
\end{pmatrix}, \quad \bar{\sigma}^\mu = \begin{pmatrix}
1_2 \\
\vec{\sigma} \\
\vec{\sigma}
\end{pmatrix}
\]
The \(\xi^\mu\) and \(\bar{\eta}^\mu\) are 2-dim complex vectors (also called Weyl Spinors) whose commutator anticommutes:
\[
\{\xi, \bar{\eta}\} = 0 \quad \Rightarrow \quad \{\xi, \bar{\eta}\} = -\{\xi, \bar{\eta}\}
\]
\[
\{\bar{\eta}, \bar{\eta}\} = 0 \quad \Rightarrow \quad \{\bar{\eta}, \bar{\eta}\} = -\{\bar{\eta}, \bar{\eta}\}
\]
\[
\{\xi, \bar{\eta}\} = 0 \quad \Rightarrow \quad \{\xi, \bar{\eta}\} = -\{\xi, \bar{\eta}\}
\]

Duality relations among 2-spinors:
\[
(\xi^\mu)^+ = \bar{\xi}^\mu, \quad (\bar{\xi}^\mu)^+ = \bar{\xi}^\mu
\]
\[
(\bar{\eta}^\mu)^+ = \eta^\mu, \quad (\eta^\mu)^+ = \eta^\mu
\]

Lowering and raising spinor indices:
\[
\xi^\nu = \xi^\nu_\rho \xi^\rho, \quad \xi^\nu = \xi^\nu_\rho \xi^\rho
\]
\[
\bar{\eta}^\nu = \bar{\eta}^\nu_\rho \bar{\eta}^\rho, \quad \bar{\eta}^\nu = \bar{\eta}^\nu_\rho \bar{\eta}^\rho
\]
\[
\text{with: } \quad \xi^\nu_\rho \equiv i\sigma^\nu = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\xi^\nu_\rho
\]
\[
\text{and } \quad \bar{\eta}^\nu_\rho \equiv i\sigma^\nu = -\bar{\eta}^\nu_\rho
\]

Spinor - scalar:
\[
\bar{\eta} \eta \equiv \bar{\xi}^\nu \xi^\nu = \bar{\xi}^\nu_\rho \xi^\rho = +\eta^\nu \xi^\nu\]
\[
= \eta^\nu \xi^\nu = \eta \bar{\xi}
\]
Likewise:
\[
\bar{\eta} \eta \equiv (\eta \xi)^+ = \bar{\xi}^\nu \bar{\eta}^\nu = \bar{\xi}^\nu_\rho \bar{\eta}^\rho = \bar{\eta}^\nu \xi^\nu
\]
\[
= \bar{\eta}^\nu \xi^\nu = \bar{\eta} \xi
\]

Note also:
\[
M \vec{\sigma}^\mu M^+ = \lambda^\nu \vec{\sigma}^\nu
\]
\[
M^+ M^{-1} \vec{\sigma}^\mu M^{-1} = \lambda^\nu \vec{\sigma}^\nu
\]

* Lorentz transformation of Weyl Spinors and Dirac Spinors

Dirac spinor is the direct sum of two Weyl spinors \(\xi\) and \(\bar{\eta}\):
\[
\Psi = \xi \oplus \bar{\eta}
\]
with Lorentz trasform properties:
\[
\bar{\eta} = \bar{\eta} \quad \Rightarrow \quad \bar{\eta} = \bar{\eta}
\]
\[
\bar{\eta} = \bar{\eta} \quad \Rightarrow \quad \bar{\eta} = \bar{\eta}
\]
\[
\bar{\eta} = \bar{\eta} \quad \Rightarrow \quad \bar{\eta} = \bar{\eta}
\]
\[
(\bar{\xi}^\nu)_\rho = (\bar{\xi}^\nu)_\rho \quad \Rightarrow \quad (\bar{\xi}^\nu)_\rho = (\bar{\xi}^\nu)_\rho
\]
\[
(\bar{\eta}^\nu)_\rho = (\bar{\eta}^\nu)_\rho \quad \Rightarrow \quad (\bar{\eta}^\nu)_\rho = (\bar{\eta}^\nu)_\rho
\]
\[
(\bar{\xi}^\nu)_\rho = (\bar{\xi}^\nu)_\rho \quad \Rightarrow \quad (\bar{\xi}^\nu)_\rho = (\bar{\xi}^\nu)_\rho
\]
\[
\text{with } \quad M \in SL(2,\mathbb{C})
\]

* Lorentz transformation of Dirac Spinor

In Matrix Form:
\[
\Psi = \begin{pmatrix} \xi \\ \bar{\eta} \end{pmatrix} \quad \Rightarrow \quad \Psi^* = \begin{pmatrix} M \xi \\ M^+ \bar{\eta} \end{pmatrix}
\]
6.4. Symmetries of the Lagrangian:

6.4.1. Global Symmetry in QFT:

A global symmetry of the CM fields, if induced by a Lie group with generators $T_a$, is:

$$\phi(x) \rightarrow \phi'(x) = e^{-2i\theta T_a} \phi(x) \quad (6.4.1)$$

where

- $\phi(x)$ is a field
- $T_a$ are generators of the Lie group

According to relation

$$U(\theta, \mathbf{r}) = e^{-i2\theta \mathbf{T}_a}$$

and

$$\theta \neq \theta(x')$$

Assume $\theta_a$ is small $\theta_a \ll 1$

$$U(\theta_a) = (1 + i\theta_a \mathbf{T}_a) \quad \theta_a \ll 1$$

Thus (6.4.1)

$$\phi_a(x) \rightarrow \phi_a'(x) = \phi_a(x) + i\theta_a \mathbf{T}_a \frac{\partial \phi_a}{\partial x_k} \mathbf{T}_k$$

$$\equiv \phi_a(x) + \delta \phi_a(x)$$

with:

$$\delta \phi_a = -i \theta_a \mathbf{T}_a \frac{\partial \phi_a}{\partial x_k} \mathbf{T}_k$$

If $\mathcal{L}(\phi_a, \bar{\phi}_a)$ is invariant under this transformation,

$$\Rightarrow \delta \mathcal{L} = 0$$

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a + \frac{\partial \mathcal{L}}{\partial \bar{\phi}_a} \delta \bar{\phi}_a \mathbf{T}_a = 0$$

E-Lagrange = 0

$$\Rightarrow \delta \mathcal{L} = 0$$

$$\Rightarrow \mathcal{L} \left( \frac{\partial \phi_a}{\partial \bar{\phi}_a} \right) = 0$$

$$= \int \frac{\partial \phi_a}{\partial \bar{\phi}_a} \left( T^a \frac{\partial \phi_a}{\partial \bar{\phi}_a} \right) = 0$$

$$\Rightarrow \mathcal{L} \left( \frac{\partial \phi_a}{\partial \bar{\phi}_a} \right) = 0$$

$$\Rightarrow \mathcal{L} = T^a \frac{\partial \phi_a}{\partial \bar{\phi}_a}$$

$$\Rightarrow \mathcal{L} \left( \frac{\partial \phi_a}{\partial \bar{\phi}_a} \right) = 0$$

but:

$$\mathbf{J}^a = 0$$

: 4-current density
\[ J^\mu = \frac{\partial L}{\partial (\partial_\mu \phi)} \]

\[ \partial_\mu J^\mu = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu \mathcal{J}^\mu = 0 \]

Conserved charge:
\[ \left( \mathcal{J} \cdot d\mathbf{s} \right) \]

Gauge law:
\[ = \int \mathcal{J} \cdot ds \]
\[ \Rightarrow \int \mathcal{J} \cdot ds = -\frac{d}{dt} \int \mathcal{E} d\mathbf{x} \]

Choosing a surface [by our gauge freedom] to go to \( \infty \):
\[ \int \mathcal{J} \cdot ds \rightarrow 0 \quad s \rightarrow \infty \]

\[ \Rightarrow \frac{d\mathcal{Q}}{dt} = 0 ; \quad \{ Q \equiv \int \mathcal{E} d\mathbf{x} \geq \int \mathcal{J} \cdot d\mathbf{x} \} \]

I.e., invariance of \( \mathcal{L} \) under Global trans.

\[ \rightarrow \] Conserve current & charge.

\[ \rightarrow \] \( \mathcal{L} \) is said to have "Global Symmetry".

5.5. Gauging QED & QCD

- Free \( \phi \) Lagrangian:
\[ \mathcal{L}_D = \overline{\psi} \left( \gamma^\mu \partial_\mu - m \right) \psi \]

- Under Global phase gauge trans.
\[ \psi \rightarrow \psi' = e^{i\theta} \psi \]

Invariance of \( \mathcal{L}_D \) results in conserved current & charge.

- Consider local phase trans:
\[ \psi(\theta) \rightarrow \psi' = e^{i\theta} \psi \]
\[ \overline{\psi} \rightarrow \overline{\psi}' = e^{-i\theta} \overline{\psi} \]

\[ \mathcal{L}_D \rightarrow \mathcal{L}'_D = \mathcal{L}_D + \delta \mathcal{L}_D \]
\[ = \overline{\psi} \left( \gamma^\mu \partial_\mu - m \right) \psi + \overline{\psi} \left( \gamma^\mu A^\mu \right) + \gamma_5 \eta \overline{\psi} \psi \]

To obtain invariance, we need to introduce a \( A \)-potential, \( A(\phi) \), that can absorb such term involving \( \partial_\mu \phi \).

From our knowledge of Maxwell's eq & related shedding eq.invarianta, we know that this \( A \)-vector must be the & EM potential with:

\[ A(\phi) \rightarrow A'(\phi) = A(\phi) + \partial_\mu \phi(x) \]

Invoking Minimal substitution:

\[ \partial_\mu \rightarrow D_\mu \equiv \partial_\mu - ie A_\mu \]

- Covariant derivative, \( D_\mu \psi \), transform just like field \( \psi \), under local (gauge) p.trans.

Proof:
\[ D_\mu \psi \rightarrow D' \psi' = \left( \partial_\mu - ie A_\mu \right) \psi' \]
\[ = \left( \partial_\mu - e A_\mu \right) \psi' \]
\[ = e^{\gamma_5 \partial_\mu} \overline{\psi} \psi \]

Note:
\[ \psi \rightarrow \psi' = e^{-i\theta} \psi \]

\[ \overline{\psi} \rightarrow \overline{\psi}' = e^{i\theta} \overline{\psi} \]

\[ \overline{\psi} D_\mu \psi \] is invariant just like \( \overline{\psi} \psi \), both are scalars in that sense.

- Gauge invariant lagrangian reads [for QED]
\[ \mathcal{L}_{QED} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \overline{\psi} \left( \gamma^\mu \partial_\mu - m \right) \psi \]
\[ - \frac{1}{2} \left( \partial_\mu A^\mu \right)^2 \]
\[ \mathcal{L} = \frac{1}{2} F_{\mu \nu} F^{\mu \nu} \]  

Free photon Lagrangian.

5.5.1. SU(2) gauge trans.

5.5.3. QCD: \( \mathcal{L} \) invariant under SU(3) gauge group:

- Fundamental objects of non-abelian SU(3), analogous to \( e \) in QED, are Coloured quarks:
  \[ \psi(x) = \begin{pmatrix} \psi_u \\
\psi_d \\
\psi_s \end{pmatrix} \]

- There is no \( Y-Y \) interaction (\( * A \) \( ^\mu A ^\mu \)) since theory is abelian (mathematically). Physically it means that \( Y \)s can be seen observed free propagating. [compare to Gluons, which do not exist as free particles]

- There objects are trans. by \( 3 \times 3 \) SU(3) matrix.

\[
\begin{align*}
\mathcal{L}(x) & \quad \rightarrow \quad \mathcal{L}(x) = U(x) \psi(x) \\
\mathcal{L}(x) & \quad \rightarrow \quad \overline{\mathcal{L}}(x) = \overline{U}(x) \mathcal{L}(x) U^\dagger(x)
\end{align*}
\]

\[ U(x) \in SU(3) \iff \begin{cases} 
\det[U(x)] = 1 \\
U U^\dagger = U^\dagger U = I_3 
\end{cases} \]

- Generally:

\( \rightarrow SU(N) \) has \((N^2-1)\) generators.

Proof:

- A matrix \( M \in SU(N) \) has \( N \times N = N^2 \) elements

- Being complex matrix \( 2N^2 \) elements (dof)

- \( M \) is unitary: \( N^2 \) constraints

- \( M \) has \( \det = 1 \) \( \Rightarrow \) 1 dof

- \#dof = \( 2N^2 - N^2 - 1 = (N^2 - 1) \).

- These dof can be taken as rotation angles, \( \theta_{ij} \), each associated with a generator \( T^a \).

- General form of \( U(x) \):

\[ U(x) = e^{i \omega_a(x) T^a} \]

- SU(3) has 8 generators, so one needs 8 new potent Fields to "frick in" the non-inv. terms resulting from trans.

- Trans. of spinor triplet:

\[ \psi(x) \rightarrow \psi'(x) = e^{i \omega_a(x) T^a} \overline{\psi(x)} \]

\[ \Rightarrow \psi(x) = e^{i \omega_a(x) T^a} \overline{\psi(x)} \]

\[ \Rightarrow \overline{\psi(x)} = -i \omega_a(x) T^a \psi(x) \]
where gauge trans. of $A_\mu$ is:

\[
A_\mu \rightarrow A'_\mu = U A_\mu U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1}
\]

Thus:

\[
D_\mu \Psi \rightarrow D_\mu' \Psi = (\partial_\mu - ig A_\mu) (U \Psi) = U (\partial_\mu - ig A_\mu) \Psi
\]

\[
= U \left( \partial_\mu - g (A_\mu U^{-1} + g (\partial_\mu U^{-1}) \right) \Psi)
\]

\[
= U \left( \partial_\mu - g A_\mu \right) \Psi
\]

\[
= U (D_\mu \Psi)
\]

i.e., D_\mu trans. just like \( \Psi \).

\[ \overrightarrow{D \Psi} \text{ is invariant } \Rightarrow \overrightarrow{D' \Psi} \text{ is also.} \]

QCD (non-abelian) field strength tensor:

\[ F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu - g [A_\mu, A_\nu] \]

Taking into account the different Gluons (gauge vectors), $A_\mu$ is an \textit{isovector}:

\[
\begin{align*}
A_\mu &= A_{\mu a} T^a \\
F_{\mu \nu} &= F_{\mu \nu a} T^a
\end{align*}
\]

\[
F_{\mu \nu a} = \partial_\mu A_{\nu a} - \partial_\nu A_{\mu a} + g F_{\nu c a b} A_{\mu b} A_{\nu c}
\]

where

\[
\partial_\mu T^a = 0, \quad [A_{\mu a}, A_{\nu b}] = 0
\]

In terms of tensors & vectors:

\[
F_{\mu \nu} = D_\mu A_\nu - D_\nu A_\mu + g A_{\mu a} \times A_{\nu b}
\]

Generators of rotations of Gluons, are

\[ [A_{\mu a}, A_{\nu b}] = \delta_{\nu a} A_{\mu b} - \delta_{\mu b} A_{\nu a} \]

Free Dirac Lagrangian:

\[
L_D = \overline{\Psi} \gamma^\mu \partial_\mu \Psi \; \text{; ignoring other terms.}
\]

Under SU(3),

\[
L_D \rightarrow L_{D}^1 = \overline{\Psi} U^{\mu a} \gamma^\mu U_a \Psi
\]

\[
= (\overline{\Psi} U^{\mu a} \gamma^\mu U_a ) U^\dagger \Psi U^{-1}
\]

\[
= \overline{\Psi} \gamma^\mu U^{\mu a} \gamma^\mu U_a \Psi
\]

Introduce \textit{compensating field through covariant derivatives:}

\[
\partial_\mu \rightarrow \partial'_\mu = \partial_\mu - ig A_\mu
\]
1. Rotation matrix expansions:

For $SU(N)$ group, we have

$$U(\theta, \phi) = e^{-i T \cdot \Phi}$$

Euler formula:

$$e^{-i T \cdot \Phi} = \cos[ T \cdot \Phi] - i \sin[ T \cdot \Phi]$$

Taylor expansion:

$$e^{-i T \cdot \Phi} = \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} (T \cdot \Phi)^k$$

Proposition: If $(T \cdot \Phi)^2 = \lambda \text{ then}$

$$\begin{cases} \cos[T \cdot \Phi] = (T \cdot \Phi)^2 \cos(\phi) \\ \sin[T \cdot \Phi] = T \cdot \Phi \sin(\phi) \end{cases}$$

- Invariant Lagrangian under $SU(3)$ infinity gauge transform:

$$L_{\text{acp}} = -\frac{1}{4} F_{\mu}^{\alpha} F^{\mu\alpha} + \bar{\psi} (i \gamma^\mu D^\mu) \psi$$

- Interaction Terms:

$$V(\psi, \bar{\psi} \sigma A_{\mu}, \psi)$$

Self-interaction